THE STRUCTURAL EFFECTS OF CYCLIC THERMAL LOADING ON A FREE ELASTOPLASTIC PLATE

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SUMMARY

The principal object of this research was to investigate the structural behavior of a free, elastic-perfectly platic plate subjected to cyclic thermal loading on its surfaces and to develop a method of predicting the resulting behavior. The study concludes that a free plate can either deform elastically, shakedown to an elastic state, yield in an alternate plasticity mode of behavior or yield in an incremental collapse mode of behavior when subjected to cyclic thermal loading. A method of predicting which mode of behavior will result for any given heating condition has been formulated. Details of the prediction procedure are presented for the special case of a linearly varying surface temperature.

CHAPTER I

INTRODUCTION

Definition of the Problem

This research consists of a study of the structural behavior of a free, elastic-perfectly plastic plate subjected to cyclic thermal loading. The plate, whose thickness is small relative to its face dimensions, is periodically heated on its surfaces so that the temperature varies only with distance from and normal to the mid-plane of the plate. Its Young's modulus, Poisson's ratio, coefficient of expansion and thermal diffusivity are assumed constant, and the yield stress in tension and compression are assumed equal. Two cases are considered, one having the yield stress remain constant and the other considering the additional effect of having the yield stress linearly decrease with temperature. The simplifying assumption that all particles reach their maximum and minimum stress levels simultaneously (simultaneous loading) will not be assumed; therefore, the frequency of the heating cycle is not restricted in any way. Neither anisotropic material nor bimetallic plates are taken up in this investigation, and creep and metallurgical changes in the material at elevated temperatures are ignored.

Review of the Literature

In engineering practice, many structures are frequently subjected to conditions in which a variation in temperature occurs during their service life. Also, since greater demands are presently being placed on our structural materials, not only in the space industry where increased emphasis is toward reducing size and weight, but in all fields of engineering where consideration of economy alone has made it necessary that materials be used more efficiently, it has become desirable to take advantage of inelastic design. Even though many solutions exist in the literature for thermal stresses where inelastic behavior is considered, this has often been ignored in design practice. This is because in most practical applications dealing with ductile materials excessive thermal stresses will be relieved through plastic flow. However, this is not true when the variation in temperature is cyclic in nature. When thermal cycling occurs, the stresses which were relieved through plastic flow in one direction in one half of the cycle may reappear in the other direction in the next half cycle as a direct result of the previous plastic flow. Hence, when considering inelastic behavior in connection with cyclic temperatures, it becomes desirable to be able to predict the structural behavior which will result. This particular area of structural behavior, which, due to its cyclic nature, has to be considered a fatigue problem, has received comparatively little attention since most fatigue testing has been concerned with 1,000 or more cycles to failure; whereas, the present problem is concerned with only the first few cycles.

When a structure is subjected to cyclic thermal loading, Parkes [1,2]* has found that the resulting thermal stresses may cause any one

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*Numbers in brackets refer to references listed in the Bibliography.

of four different types of structural deformation to occur. The four types of behavior are as follows:

1. *Permanent Elasticity*

The thermal stresses may remain below the yield stress of the material. In this case the structure will return to its original shape at the end of each cycle.

2. *Shakedown to an Elastic State*

The thermal stresses may exceed the yield stress during the first temperature cycle, producing plastic deformation, but then shakedown into the elastic region in the second cycle and remain elastic thereafter. This final elastic configuration, though different from the initial one, is thereafter the same at the end of each temperature cycle.

3. *Alternate Plasticity*

The thermal stresses may exceed the yield stress in each cycle of loading, yet producing reversed plastic deformation so that the configuration is the same at the end of each cycle. However, the plastic strain produced during the first cycle might not be fully recovered.

4. *Incremental Collapse*

The thermal stresses may again exceed the yield stress in each temperature cycle, but in this case produce plastic strains which have a nonzero net value for each cycle of loading. The configuration will thus progressively change shape due to this incremental straining.

The first two types of structural behavior, elastic behavior and shakedown, do not produce unacceptable structural effects since the useful life of the structure is generally very long. Under alternating plasticity, even though excessive structural deformation is not produced, an early fatigue failure may result. In the fourth case, incremental collapse, large deformation can result in a limited number of cycles; therefore, this type of behavior is generally structurally unacceptable. It may be tolerable only if the number of cycles is very small during the life of the structure.

Parkes has illustrated these four types of structural behavior by considering a structure which consists of two bars which are subjected to a constant load, constrained to remain of equal length, and initially at a uniform temperature conveniently denoted by 0 (Figure 1). One bar was subjected to a temperature cycle 0 - T - 0 - ..., where T is some higher temperature above the initial temperature 0, while the other bar remained at the initial temperature 0. He showed that, depending on the values of the load $(\frac{W}{2A})$, σ_0, σ_T, and T, one of the four types of behavior will occur, as can be seen in Figure 1.

Parkes more fully analyzed the structural behavior of the two-bar structure in another paper [3], mainly to determine the influence of the yield point/temperature relation on its behavior. In so doing he also considered the effect of heat conduction from one bar to the other. The structure was the same as before with two identical bars except in this case no constant load was applied. The temperature cycle was as follows:

<u>Temperature Cycle</u>

Bar 1: $0 \rightarrow T \rightarrow 0 \rightarrow$

Bar 2: $0 \rightarrow 0 \rightarrow 0 \rightarrow$

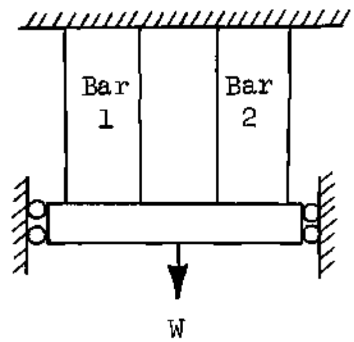
A = Cross-sectional area of each bar

E = Young's modulus

α = Coefficient of thermal expansion

σ_o = Yield stress at temperature 0

σ_T = Yield stress at temperature T



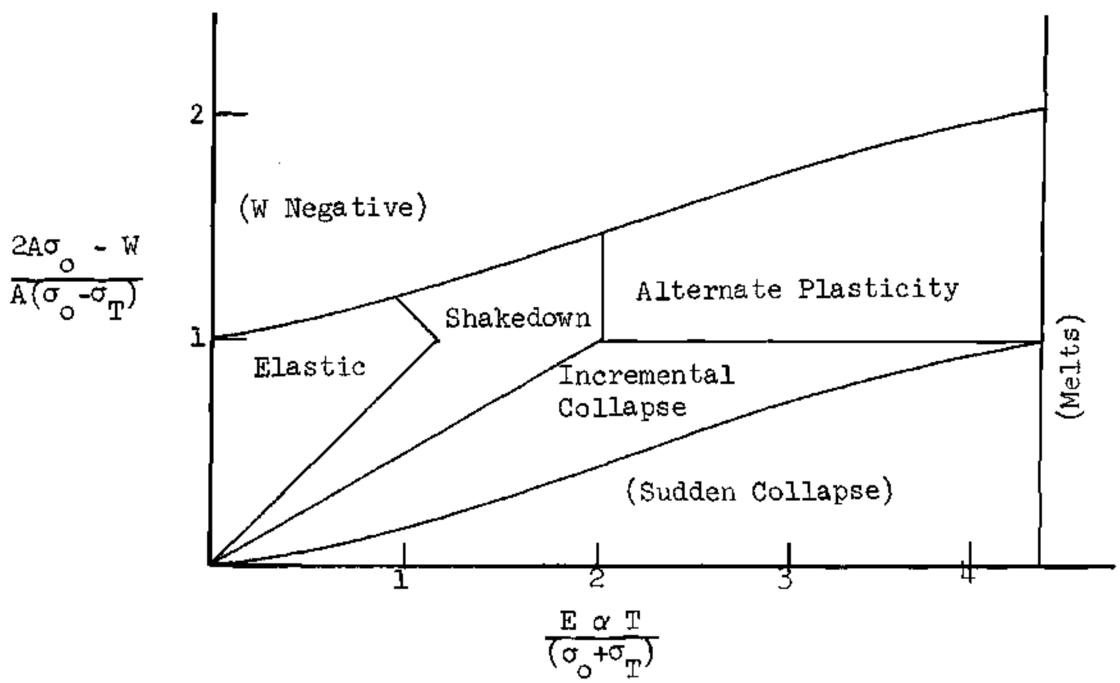


Figure 1. Zones of Behavior of Two-Bar Structure

$$\text{Bar 1:} \quad 0 - T - T - 0 - 0$$

$$\text{Bar 2:} \quad 0 - 0 - \delta T - \delta T - 0$$

where $(0<\delta\leq 1)$. The two bars would experience such a temperature cycle if bar 1 represented the outside of a structure, bar 2 represented the inside, and the temperature of the external environment varied periodically from temperature 0 to temperature T. In the structural behavior of this model, Parkes has shown that alternating plasticity would hardly ever occur. However, this is only true when considering two bars of equal areas as Parkes has here. If one bar's area is larger than the other, then alternate plasticity would occur more often and incremental collapse less often.

The basic structural analysis which Parkes has clearly illustrated, by use of the two-bar assembly, in the previously mentioned references has been extended and successfully applied to many different structures which are subjected to cyclic thermal loading. Parkes himself [4,5] applied a similar analysis to an aircraft wing subjected to cycles of kinetic heating as well as a constant bending moment. He considered one web of the cellular construction of the wing (I-section) to be represented by one bar and the wing's upper and lower skin area by the other bar. However, in this case the bending moment produced an initial linear stress distribution through the web section. In reference [4], Parkes assumed the yield stress to be constant and all four types of structural behavior was found to occur, but incremental collapse was rather rare. When the yield stress was assumed to decrease with temperature, as was done in reference [5], it was found that incremental collapse will occur more often.

Payne [6] applied the two-bar analysis to a T-section beam where the flange was subjected to a temperature cycle 0 - T - 0 - T - ..., while the entire web was assumed to remain at temperature 0. He considered an elastic-perfectly plastic stress-strain relation, no load, and one case where the yield stress was constant and another case where the yield stress decreased with temperature. The analysis of the T-section was different from the two-bar analysis in that a linear stress distribution exists in the web of the T-section resulting in part of the web remaining elastic throughout the temperature cycle; whereas, each component of the two-bar structure is always at some uniform stress level. This led Payne to the conclusion that a T-section can have elastic, shakedown, or alternate plasticity types of deformation, but since part of the web remains elastic, incremental collapse cannot occur, even in the case of a reduction of yield stress with temperature.

Progressive growth has been observed by Weil and Rapasky [7] in pressure vessels when subjected to cyclic thermal stresses. Miller [8] analyzed this progressive expansion by considering a thin-walled pressure vessel with a cyclic uniform heat flux through its walls as well as with a uniform heat generation within its walls. The analysis requires a very slow heating cycle so that simultaneous loading occurs throughout the thickness and so that, upon cooling, the entire wall returns to a uniform temperature; i.e., similar to the action of a two-bar structure when there is no heat conduction from one bar to the other. Miller presented some limiting values of stresses to avoid incremental collapse and also showed that the approximation by use of a two-bar analysis was at best 100 per cent in error.

As in all other areas of stress analysis where plates are encountered, an extensive investigation of thermal stress in plates has been developed. The analysis has usually been based on the assumption of perfectly elastic behavior [9]; whereas, inelastic behavior has only recently been considered [10-14]. In references [10-12] the material was assumed to be elastic-perfectly plastic and all mechanical and thermal properties of the material were assumed independent of temperature. Both Weiner [10] and Yuksel [11] analyzed the problem of a free plate subjected to a very slowly varying temperature on the one surface. Weiner assumed the opposite surface and edges to be perfectly insulated; whereas, Yuksel held the opposite face at a constant temperature. Since the surface temperature did not vary too rapidly, the temperature distribution through the plate could be expressed as a very simple quadratic function, thus producing a symmetric stress distribution about the median plane. Also, since all points of the plate reached their maximum load simultaneously, the stress computations were greatly simplified. For different maximum surface temperatures, it was found that the stress distribution was either elastic, elastic-plastic with two plastic regions, or elastic-plastic with three plastic regions.

Even though both Weiner and Yuksel solved essentially the same problem of slowly heating the surface of a plate and arrived at the same conclusions, Yuksel actually considered a plate subjected to a slow harmonically varying temperature on one face. He did not stop with just the heating portion but considered a periodical variation in the surface temperature in the form of a sine function. Yuksel obtained the same results in the cooling half cycle as in the heating half cycle and concluded that equal plastic regions developed successively in compression and then in tension.

Parkes [12] considered the thermal stresses of a free elastic-perfectly plastic bar subjected to a sudden change in temperature on two opposite faces and insulated on the other surfaces. His analysis could be extended to the case of quenching a uniformly heated plate with infinite heat transfer coefficients on both surfaces. He found that the number of plastic zones which occur depends on the ratio of the thermal strain (αT) to yield strain (σ_{yp}/E).

Mendelson and Spero [13] considered the elastoplastic thermal stress and strain distributions in a plate with a uniaxial temperature gradient, strain-hardening, and temperature-dependent mechanical properties. They showed, for the case of linear strain-hardening and simultaneous loading, that the strain equation is a linear Fredholm equation of the second kind, and its solution can be obtained by any of the standard methods for solving such equations and can in many cases be obtained in closed form. For the general case of nonlinear strain-hardening and/or for the case of nonsimultaneous loading, Mendelson and Spero obtained a solution by successive approximations.

In plastic flow problems where nonsimultaneous loading occurs, the final strain can be determined only by considering the complete history of stress and not merely from the final stress distribution. Manson [14] and Mendelson [15] discussed the incremental theory of plasticity, which makes use of the method of successive-approximation, and its application to thermal stress problems. Incremental theory takes into account the stress history of the material during loading and unloading; therefore, the final stress and strain can be determined from it. An application of the incremental theory of plasticity has recently been given by Murphy [16] in solving general two-dimensional thermal stress problems in the elastic-plastic range, particularly for application to rectangular plates.

Object of Investigation

As can be seen from the literature review, a number of investigations have recently been made which were concerned with inelastic behavior due to thermal stress distributions in a plate or bar subjected to a change in temperature on one or more surfaces. However, Yuksel [11] was the only one concerned with cyclic effects. He considered a harmonically varying surface temperature and his analysis restricted the frequency to very low values so that simultaneous loading would result. His only conclusion was that the plastic zones also behave in a periodical manner, yielding in compression and then in tension by equal amounts in each cycle. His plastic-flow analysis of the thermally stressed plate is therefore only a preliminary step in the investigation of cyclic thermal loading.

Referring to the two-bar structural analysis which Parkes [1,2] developed, it was found that four types of structural deformation may occur when the structure is subjected to cyclic thermal loading: permanent elasticity, shakedown to an elastic state, alternate plasticity, and incremental collapse. This analysis, which can be used to predict the occurrence of fatigue failure due to cyclic thermal stresses in a two-bar structure, has not at present been extended to the point that a continuous medium, which would contain a stress and strain distribution resulting from a cyclic temperature distribution through the medium, can be analyzed. The object of this research is to investigate a plate subjected to cyclic thermal loading to see if these four types of structural behavior will occur. A means of analyzing and predicting the various modes of behavior which result will be developed in order to determine the exact conditions for their occurrence.

Investigation Procedure

The nonlinear temperature distribution produced by cyclic heating both faces of the plate is a problem in heat conduction and may be obtained by one of the known methods of heat transfer depending upon the particular conditions involved; it was obtained here by a finite difference solution of the diffusion equation. The thermal stresses in each element of the plate resulting from this temperature distribution were calculated in an iteration procedure by using the incremental theory of plasticity developed by Manson [14] and by a numerical integration of the thermal stress equation for plates. A digital computer program was developed to perform the above numerical solution.

The effect of several variables on the structural behavior of the plate was investigated by incorporating them into the above numerical solution. The variables were:

1. Temperature range of the surface of the plate.

2. Rate of heating the surface of the plate.

3. Frequency and shape of the temperature cycle.

4. Yield stress-temperature relationship.

A thorough understanding as to the various structural behavior which can result and how they are affected by the values of the variables was necessary before a method for analytically predicting the modes of behavior was possible. This required a thorough knowledge of thermal stress analysis as well as the use of a computer. It was felt that if an approximation could be made of this analysis by approximating the plate by some system of bars and then studying the effect of the above variables on this structural model, a better understanding of the behavior of the plate could be obtained. Therefore a model was found and a method of analysis was developed for predicting several of the modes of behavior of the plate, one which can be solved in closed form without the use of the method of successive approximations, incremental theory of plasticity, and numerical integration and also without the use of a computer.

CHAPTER II

INCREMENTAL THEORY OF PLASTICITY

General

The theory of plasticity which deals with the behavior of ductile materials beyond the elastic range, unlike the classic theory of elasticity, is a comparatively new field beginning at about the turn of the century. Plastic flow analysis is essentially divided into two different sets of theories. The first is concerned with deciding when yielding will occur in the case of multiaxial stresses and is called the yield criteria. Next, to be able to describe the behavior of the material when yielding is occurring; i.e., the relationship between stress and plastic strain, theories concerning the flow rules were developed. This development is different from that in the elastic range in that the relationship between stress and plastic strain is generally nonlinear and that the strain state is not uniquely determined by the stress state for a general loading history but depends on the manner in which the stress state is attained.

This dependence of the plastic strain on the history of loading can be seen by considering a uniaxial tensile test as illustrated by Figure 2. If the material is loaded beyond the initial yield point A, say to point B, and then unloaded, say to point C, it is readily seen that even though the same stress state and elastic strain state exist at point A as at point C, their plastic strain states differ. This is

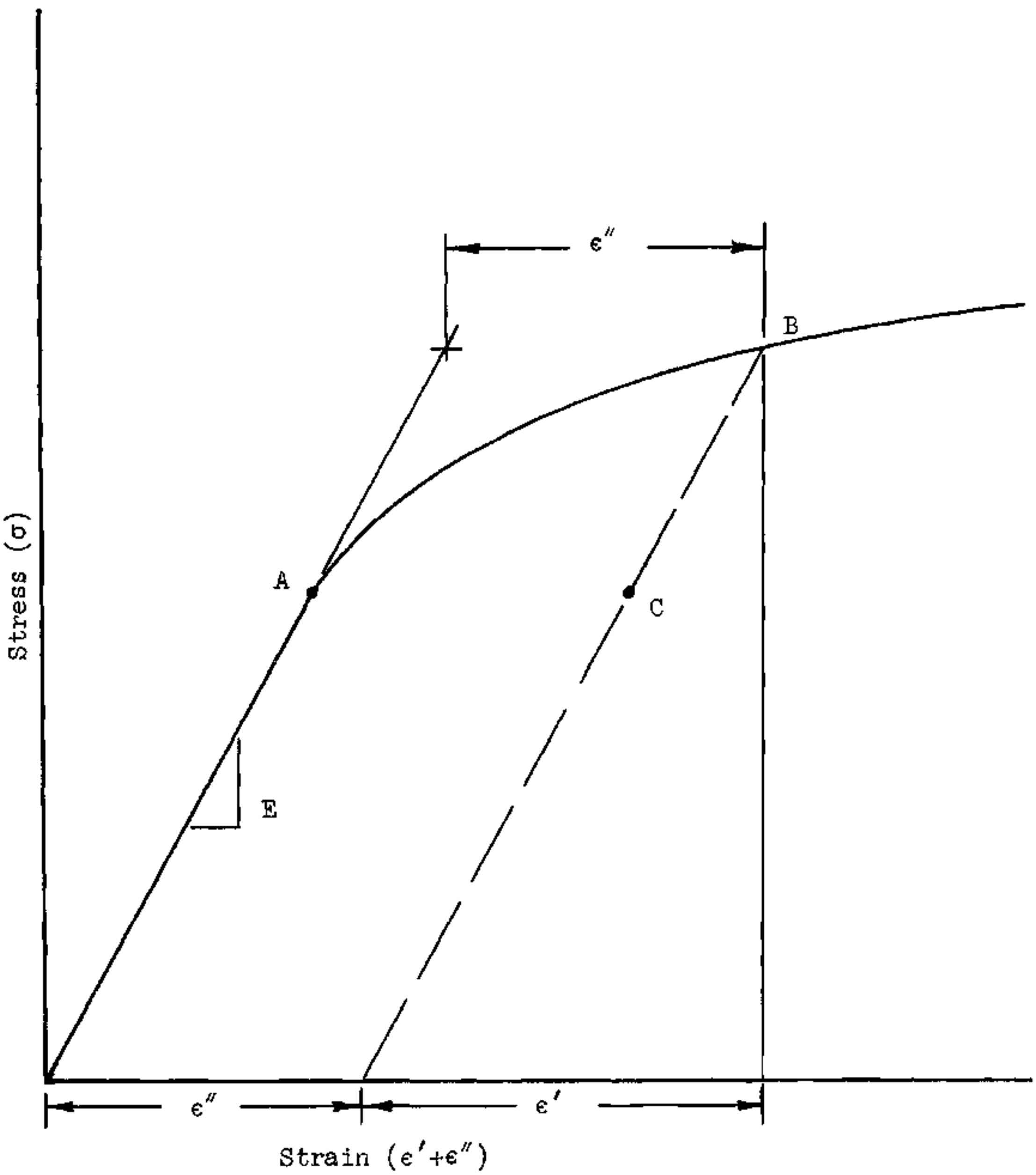


Figure 2. Uniaxial Stress-Strain Curve

due to the fact that unloading affects only the elastic strain and not the plastic strain. Since a unique relationship no longer exist between a given stress and strain, the final plastic strain can be found only by considering the complete history of stress and not merely from the value of the final stress.

In general there is no unique relationship between stress and plastic strain; however, there is one type of loading history for which the final stress state describes the entire loading path and the plastic strain state is therefore uniquely defined by it. This type of loading is called proportional loading. For proportional loading to exist, the ratios of the principal stresses must remain constant and no unloading may occur.

The plasticity theory in which the total plastic strain is obtained from the value of the final stress state is called deformation theory and can be used only for the case of proportional loading. In general, due to the above illustrated nonuniqueness, an increment procedure must be used which is called incremental theory of plasticity. Incremental theory relates the increments of plastic strain to the stress throughout the loading history and then sums the increments of plastic strain to obtain the total plastic strain. The only limitation on the size of increments is that the loading occurring during the increment must be a proportional type loading.

Basic Theoretical Equations

The general case of elastic stress and strain distribution in three dimensions can be determined from four sets of equations; the compatibility equations, the stress-strain relations, and the boundary conditions. Mendelson [15] has made use of a similar set of equations in solving an elastoplastic thermal stress problem. The only difference in the derivation of these equations and the general elasticity equations is in the relationship between stress and strain.

The total strain at each point in the material is made up of strain due to stress brought about by the continuity of the material and strain due to thermal expansion which is proportional to the temperature rise T. The temperature change affects only the normal strain since the thermal expansion at any point in an isotropic material is the same in all directions. This normal strain is equal to αT where α is the coefficient of linear thermal expansion. The strain due to stress can be divided into an elastic strain, ε'_{ij}, plus a plastic strain, ε''_{ij}. The total strain is expressed therefore in the form

$$\varepsilon_{ij} = \varepsilon'_{ij} + \varepsilon''_{ij} + \delta_{ij}\alpha T$$

The elastic parts of the total strains are related to the stresses by Hooke's Law

$$\varepsilon'_{ij} = \frac{1 + \nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{kk}$$

The plastic parts are related to the stresses by more complicated functions which will be handled separately; therefore, when substitution is made only for the elastic part, the elastoplastic thermal stress-strain relations become

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] + \alpha T + \varepsilon''_x + \Delta\varepsilon''_x$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)] + \alpha T + \varepsilon''_y + \Delta\varepsilon''_y$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] + \alpha T + \varepsilon''_z + \Delta\varepsilon''_z \qquad (1)$$

$$\varepsilon_{xy} = \frac{1 + \nu}{E}\sigma_{xy} + \varepsilon''_{xy} + \Delta\varepsilon''_{xy}$$

$$\varepsilon_{yz} = \frac{1 + \nu}{E}\sigma_{yz} + \varepsilon''_{yz} + \Delta\varepsilon''_{yz}$$

$$\varepsilon_{zx} = \frac{1 + \nu}{E}\sigma_{zx} + \varepsilon''_{zx} + \Delta\varepsilon''_{zx}$$

where ε''_x, ε''_y, etc., are the total accumulated plastic strains but do not include the plastic strain increments $\Delta\varepsilon''_x$, $\Delta\varepsilon''_y$, etc., due to the current increment of loading.

As discussed in the previous section, the plastic strain increments are related to the stresses through the yield criterion and the flow rules. Since the distortion energy or von Mises yield condition usually corresponds to the experimental data better and is easier to apply for general types of loading, it is the most widely accepted yield condition at the present time. According to this yield criterion, yielding occurs whenever the distortion energy in the material reaches a certain value expressed by:

$$\frac{1}{\sqrt{2}}\sqrt{(\sigma_x-\sigma_y)^2 + (\sigma_y-\sigma_z)^2 + (\sigma_z-\sigma_x)^2 + 6(\sigma_{xy}^2+\sigma_{yz}^2+\sigma_{zx}^2)} = \sigma_{yp} \qquad (2)$$

where σ_{yp} is the yield stress in an uniaxial tensile test. For convenience, the flow rules developed by Prandtl-Reuss will be used; therefore,

$$\Delta\varepsilon''_x = \frac{\Delta\varepsilon''_e}{\sigma_e}[\sigma_x - \tfrac{1}{2}(\sigma_y+\sigma_z)] \qquad (3)$$

$$\Delta\varepsilon''_y = \frac{\Delta\varepsilon''_e}{\sigma_e}[\sigma_y - \tfrac{1}{2}(\sigma_z+\sigma_x)]$$

$$\Delta\varepsilon''_z = \frac{\Delta\varepsilon''_e}{\sigma_e}[\sigma_z - \tfrac{1}{2}(\sigma_x+\sigma_y)]$$

$$\Delta\varepsilon''_{xy} = \frac{3}{2}\frac{\Delta\varepsilon''_e}{\sigma_e}\sigma_{xy}$$

$$\Delta\varepsilon''_{yz} = \frac{3}{2}\frac{\Delta\varepsilon''_e}{\sigma_e}\sigma_{yz}$$

$$\Delta\varepsilon''_{zx} = \frac{3}{2}\frac{\Delta\varepsilon''_e}{\sigma_e}\sigma_{zx}$$

where

$$\Delta\varepsilon''_e = \frac{\sqrt{2}}{3}\sqrt{(\Delta\varepsilon''_x-\Delta\varepsilon''_y)^2+(\Delta\varepsilon''_y-\Delta\varepsilon''_z)^2+(\Delta\varepsilon''_z-\Delta\varepsilon''_x)^2+6[(\Delta\varepsilon''_{xy})^2+(\Delta\varepsilon''_{yz})^2+(\Delta\varepsilon''_{zx})^2]} \qquad (4)$$

and

$$\sigma_e = \frac{1}{\sqrt{2}}\sqrt{(\sigma_x-\sigma_y)^2 + (\sigma_y-\sigma_z)^2 + (\sigma_z-\sigma_x)^2 + 6(\sigma_{xy}^2+\sigma_{yz}^2+\sigma_{zx}^2)} \tag{5}$$

$\Delta\varepsilon_e''$ is referred to as an equivalent or effective plastic strain increment and σ_e as an equivalent or effective stress. They are related to each other through the uniaxial tensile stress-strain curve. This can be seen by considering a uniaxial tensile test in the x-direction. The effective stress and effective plastic strain increment for this case become

$$\sigma_e = \sigma_x$$

and

$$\Delta\varepsilon_e'' = \Delta\varepsilon_x''$$

and yielding begins when

$$\sigma_e = \sigma_{yp}$$

This is in agreement with von Mises yield condition; for if one compares Equation (2) with Equation (5), it is seen that the effective stress is the same as the von Mises yield function. Therefore, the uniaxial stress-strain curve supplies the functional relationship between σ_e and $\Delta\varepsilon_e''$.

The equations of equilibrium

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z} = 0 \tag{6}$$

$$\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\sigma_{yz}}{\partial z} = 0$$

$$\frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z} = 0$$

and the compatibility equations in terms of strain components

$$\frac{\partial^2\varepsilon_x}{\partial y^2} + \frac{\partial^2\varepsilon_y}{\partial x^2} = 2\,\frac{\partial^2\varepsilon_{xy}}{\partial x\partial y} \tag{7}$$

$$\frac{\partial^2\varepsilon_x}{\partial z^2} + \frac{\partial^2\varepsilon_z}{\partial y^2} = 2\,\frac{\partial^2\varepsilon_{yz}}{\partial y\partial z}$$

$$\frac{\partial^2\varepsilon_z}{\partial x^2} + \frac{\partial^2\varepsilon_x}{\partial z^2} = 2\,\frac{\partial^2\varepsilon_{zx}}{\partial z\partial x}$$

$$\frac{\partial}{\partial x}\left\{- \frac{\partial\varepsilon_{yz}}{\partial x} + \frac{\partial\varepsilon_{zx}}{\partial y} + \frac{\partial\varepsilon_{xy}}{\partial z}\right\} = \frac{\partial^2\varepsilon_x}{\partial y\partial z}$$

$$\frac{\partial}{\partial y}\left\{- \frac{\partial\varepsilon_{zx}}{\partial y} \quad \frac{\partial\varepsilon_{xy}}{\partial z} \quad \frac{\partial\varepsilon_{yz}}{\partial x}\right\} = \frac{\partial^2\varepsilon_y}{\partial z\partial x}$$

$$\frac{\partial}{\partial z}\left(-\frac{\partial \varepsilon_{xy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y}\right) = \frac{\partial^2 \varepsilon_z}{\partial x \partial y}$$

are the same as those of isothermal elasticity since they are based on mechanical and geometrical considerations, respectively, and are not dependent on the existence of elasticity.

Iteration Procedure

The iteration procedure used in incremental theory of plasticity is usually referred to as a successive approximation method. When applying the method to plastic flow problems, Mendelson [15] refers to it as the method of successive elastic solutions since each iteration is essentially the solution of an elastic problem. Even though there is no unique order or procedure to use when applying the method of successive approximations, the solution of an elastoplastic thermal stress problem by use of this method can be obtained in the following manner. The loading history is divided into time increments so that the loading is actually incrementally stepped through time. If the heat transfer calculations have been made by use of a numerical method, then the same time increments can be conveniently used here.

For the first loading increment, the total accumulated plastic strains, ε_x'', ε_x'', etc., and the plastic strain increments $\Delta\varepsilon_x''$, $\Delta\varepsilon_y''$, etc., are first assumed zero. The stress-strain relations (1), equilibrium equations (6), and compatibility equations (7), along with the appropriate boundary conditions are solved by any convenient method (usually a numerical method) used for elasticity problems to obtain the first approximation for the stresses $\sigma_x^{(1)}$, $\sigma_y^{(1)}$, etc. In order to see if the stress-strain properties of the material are satisfied by this stress distribution, the effective stress $\sigma_e^{(1)}$ is now determined from Equation (5) and it is readily seen by von Mises yield condition if yielding has occurred at any points in the material. At these points of yielding the above elastic solution gives values of $\sigma_e^{(1)}$ which might correspond, for example, to Point A in Figure 3. Since the relationship between effective stress and effective plastic strain is supplied by the uni-axial stress-strain curve, the effective plastic strain increment $\Delta\varepsilon_e''^{(1)}$ is obtained by dropping from point A to point A' and reading the plastic portions of the strain. A value for the plastic strain increments $\Delta\varepsilon_x''$, $\Delta\varepsilon_y''$, etc., can now be obtained from the Prandtl-Reuss relations (3) by making use of the above effective stress and effective plastic strain increment values and also the previously obtained elastic stress approximations.

Using these values of the plastic strain increments in the above, mentioned sets of equations, the stress distribution $\sigma_x^{(2)}$, $\sigma_y^{(2)}$, etc., and the effective stress $\sigma_e^{(2)}$ are determined in the same manner as before, i.e., an elastic solution. The values of the effective stress at a point of yielding would possibly correspond to point B in Figure 3. Dropping down to point B' on the stress-strain curve, the effective plastic strain increment $\Delta\varepsilon_e''^{(2)}$ is again obtained. Now use the new stress components $\sigma_x^{(2)}$, $\sigma_y^{(2)}$, etc.,effective stress $\sigma_e^{(2)}$ and effective plastic strain increment $\Delta\varepsilon_e''^{(2)}$ in Equations (3) to obtain new values for the plastic strain components $\Delta\varepsilon_x''$, $\Delta\varepsilon_y''$, etc. By repeating the above procedure, a set of points A', B', C',..., are produced. This is continued until the effective stress value is sufficiently close to the

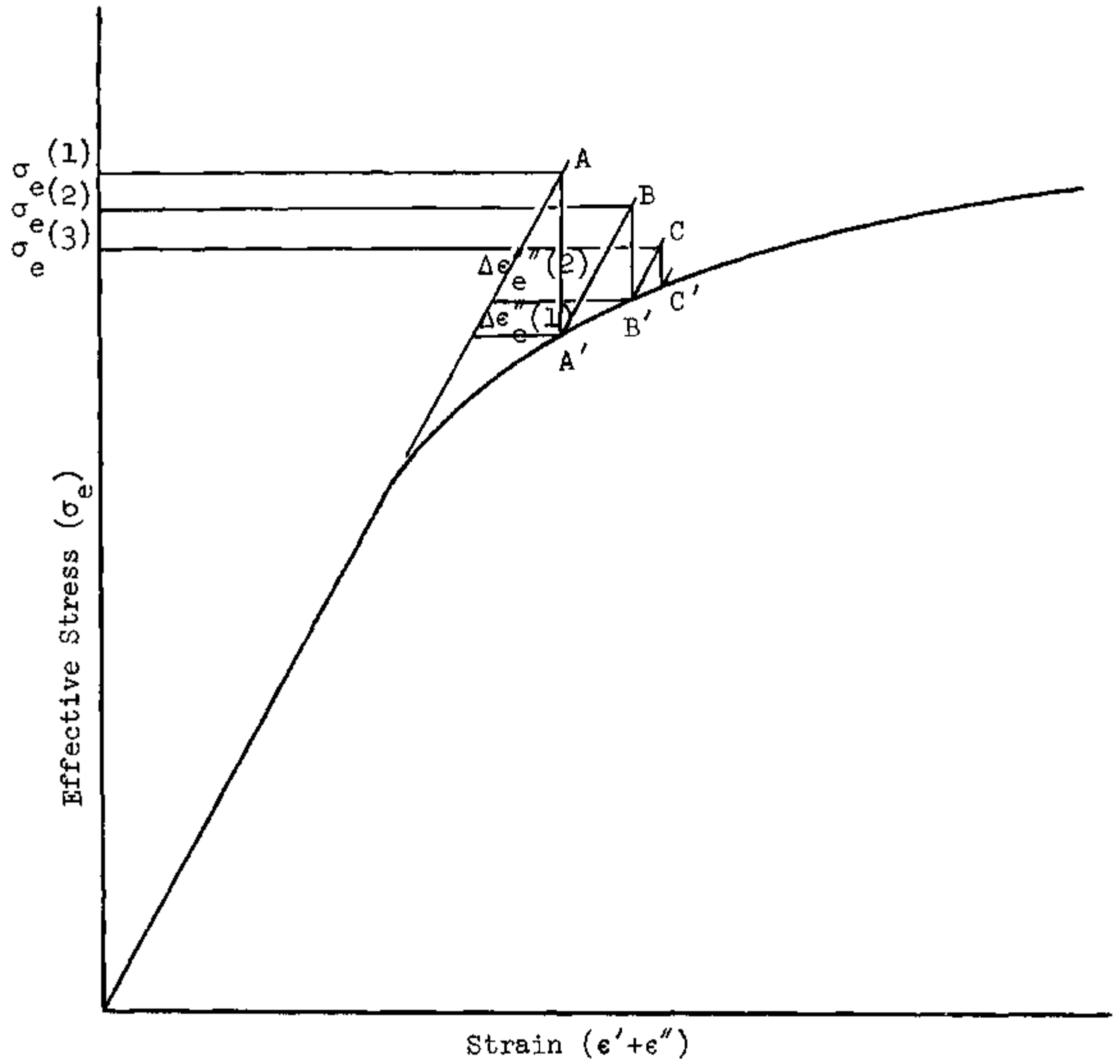


Figure 3. Typical Stress-Strain Curve

uniaxial stress-strain curve; that is, the difference between two successive values of effective stress is less than some prescribed value for each point in the material.

When a solution has been obtained by the above successive approximation method for the first increment of loading, the values of the plastic strain increments $\Delta\varepsilon_x''$, $\Delta\varepsilon_y''$, etc., are added to the accumulated plastic strains ε_x'', ε_y'', etc., and the plastic strain increments are again set to zero. For the next increment of loading or unloading, the exact same successive approximation procedure is again carried out except that now ε_x'', ε_y'', etc., are increased or decreased by the values of $\Delta\varepsilon_x''$, $\Delta\varepsilon_y''$, etc., from the previous increment of loading. In this manner the stress and strain distribution can be determined for any loading history, in particular for the nonsimultaneous loading and unloading produced by cyclic thermal stress conditions.

CHAPTER III

INELASTIC STRESS ANALYSIS OF PLATES

Formulation of the Problem

General Description

The problem investigated is that of two-dimensional plane stress in a plate of uniform thickness 2h which is small relative to the plate's face dimensions. The boundary conditions with respect to the rectangular coordinates indicated in Figure 4 are as follows.

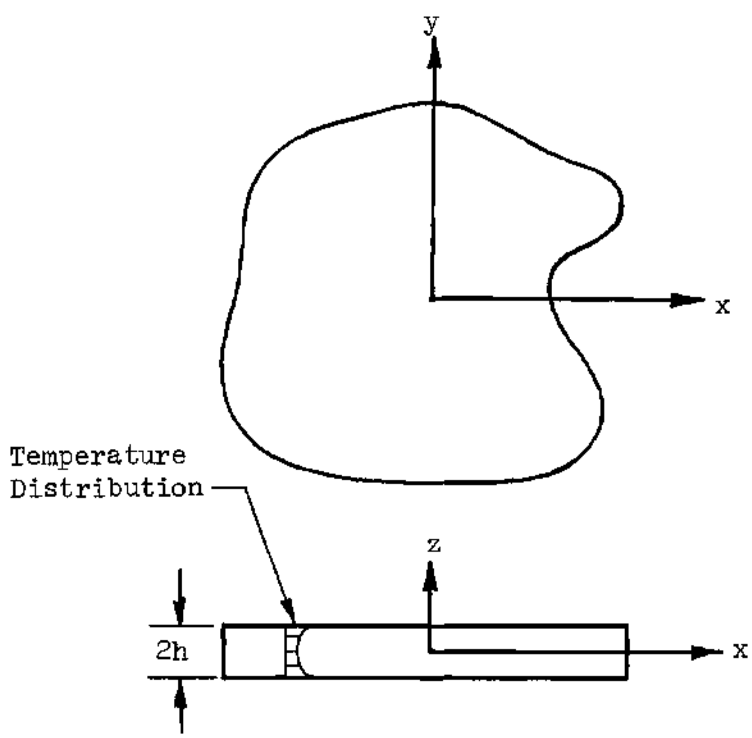


Figure 4. Plate Configuration

A periodic temperature variation on the faces $z = \pm h$ is assumed uniform across the surfaces so that the temperature varies only with distance from and normal to the mid-plane of the plate; that is, in the z-direction; the edges of the plate are assumed perfectly insulated. The plate is completely free of surface tractions and all body forces are assumed zero.

At present, little is known about how the properties of materials (such as creep, strain-hardening, metallurgical changes, etc.) vary under cyclic thermal conditions; therefore, it is generally felt that the best assumption would be to assume that the material exhibits an elastic-perfectly plastic stress-strain relationship. However, since the yield stress is known to decrease with temperature for many materials, two cases are considered, one with the yield stress remaining constant and the other having the yield stress linearly decrease with temperature. For both cases, the coefficient of thermal expansion (α), Young's modulus (E), Poisson's ratio (ν), and thermal diffusivity (a) are considered constant, and the yield stress in tension and compression are assumed equal.

An example of the variation of temperature $T(z,t)$ through the thickness is shown in Figure 5 when the surface temperature is varied with time as shown. It is assumed that periodic steady-state temperature oscillations have been established. Note should be made of the exponentially decreasing magnitude of the temperature with distance from the surfaces and of the time lag which increases exponentially with distance from the surfaces, producing nonsimultaneous loading and unloading through the plate.

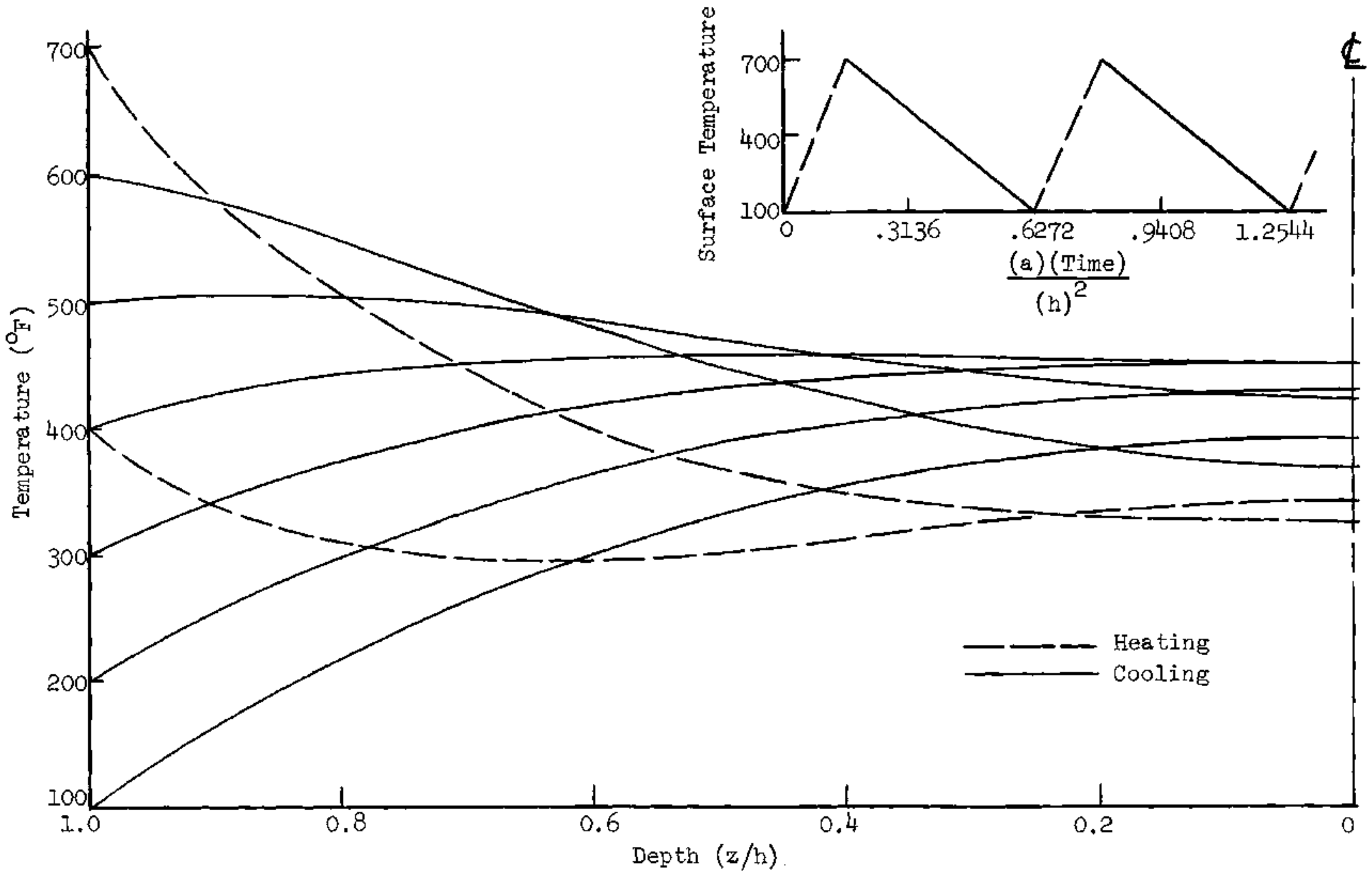


Figure 5. Variation of Temperature with Depth When the Surface Temperature Varies Periodically with Time As Shown

Basic Equations

In order to obtain an expression for the stresses for the particular cyclic elastoplastic thermal stress problem being considered here, a similar derivation to that used by Boley and Weiner [17] will be used. That is, if one considers the particular conditions of this problem, it is understandable that the assumption that

$$\sigma_x = \sigma_y = \sigma(z,t)$$

and

$$\sigma_z = \sigma_{xz} = \sigma_{yz} = \sigma_{xy} = 0$$

can reasonably be made. After the foregoing it is only necessary to show that a solution of this form will indeed satisfy the basic sets of equations and therefore is the correct solution.

It is seen immediately that the three-dimensional equilibrium equations (6) are satisfied identically by stress components of the assumed form. By substituting the stress-strain relations, Equation (1), into the compatibility Equation (7), it is seen that they are also satisfied by the assumed form of solution provided that

$$\frac{d^2}{dz^2}\left[\sigma(z,t) + \frac{E}{1-\nu}\{\alpha T(z,t)+\varepsilon''(z,t)+\Delta\varepsilon''(z,t)\}\right] = 0$$

where

$$\varepsilon''(z,t) = \varepsilon''_x(z,t) = \varepsilon''_y(z,t)$$

$$\Delta\varepsilon''(z,t) = \Delta\varepsilon''_x(z,t) = \Delta\varepsilon''_y(z,t)$$

The general solution is therefore

$$\sigma(z,t) = -\frac{E}{1-\nu}\{\alpha T(z,t)+\varepsilon''(z,t)+\Delta\varepsilon''(z,t)\} + C_1 z + C_2$$

The constants of integration C_1 and C_2 are determined from the boundary condition that the plate is completely free of surface tractions, in particular, free of surface tractions across the edges of the plate. However, if the surface tractions across the edges are zero, then for the elastic case the temperature would only vary linearly with z producing a state of zero stress throughout the plate. If the stresses are zero for the elastic case, then it follows that the plastic case would never occur.

A solution can, however, be obtained if these local effects near the plate edges are neglected and Saint-Venant's principle is applied. Saint-Vernant's principle modifies the boundary conditions by assuming that if the forces acting on a portion of the surface of an elastic body are replaced by a statically equivalent set, then only the local stress distribution is affected. The error in the solution is negligible for distances from the edges of more than approximately one plate thickness. The statically equivalent boundary conditions which replace the free edge conditions state that the resultant force and moment per unit length are zero across the edges of the plate and are expressed

$$\int_{-h}^{h} \sigma(z,t)dz = \int_{-h}^{h} \sigma(z,t)zdz = 0$$

By applying these conditions instead of the actual ones, the constants C_1 and C_2 can be found, producing the required solution

$$\sigma(z,t) = \frac{E}{1-\nu}\left[\frac{1}{2h}\int_{-h}^{h} \{\alpha T(z,t)+\varepsilon''(z,t)+\Delta\varepsilon''(z,t)\}dz\right.$$

$$+ \frac{3z}{2h^3}\int_{-h}^{h} \{\alpha T(z,t)+\varepsilon''(z,t)+\Delta\varepsilon''(z,t)\}zdz$$

$$\left. - \{\alpha T(z,t)+\varepsilon''(z,t)+\Delta\varepsilon''(z,t)\}\right]$$

which may also be written

$$\sigma(Z,t) = \frac{E}{1-\nu}\left[\frac{1}{2}\int_{-1}^{1} \alpha T(Z,t)dZ + \frac{3Z}{2}\int_{-1}^{1} \alpha T(Z,t)ZdZ - \alpha T(Z,t)\right]$$

$$+ \frac{E}{1-\nu}\left[\frac{1}{2}\int_{-1}^{1} \{\varepsilon''(Z,t)+\Delta\varepsilon''(Z,t)\}dZ\right. \tag{8}$$

$$\left. + \frac{3Z}{2}\int_{-1}^{1} \{\varepsilon''(Z,t)+\Delta\varepsilon''(Z,t)\}ZdZ - \{\varepsilon''(Z,t)+\Delta\varepsilon''(Z,t)\}\right]$$

where $Z = z/h$ is the dimensionless distance from the plate's mid-plane $Z = 0$. The original form of the solution satisfies identically the boundary condition that the faces $Z = \pm 1$ are free of surface tractions. Hence, a solution of the above equation will be the correct solution of the stated problem, satisfying the basic sets of equations which were used in its derivation including the boundary conditions, that is, at least to within the accuracy stated by Saint-Venant's principle.

Since the problem investigated, as described in the previous section, consists of a plate having a periodic temperature variation on the surfaces $Z = \pm 1$, the temperature history $T(Z,t)$ is symmetric about $Z = 0$, i.e., an even function of Z. Therefore

$$\int_{-1}^{1} T(Z,t) Z dZ = 0$$

Also since the plastic zones occur as a direct result of the temperature variation, the plastic strain, $\varepsilon''(Z,t) + \Delta\varepsilon''(Z,t)$ is also an even function of Z. The result being that

$$\int_{-1}^{1} \{\varepsilon''(Z,t) + \Delta\varepsilon''(Z,t)\} Z dZ = 0$$

Equation (8) therefore reduces to

$$\sigma(Z,t) = \frac{E}{1-\nu}\left[\frac{1}{2}\int_{-1}^{1} \alpha T(Z,t) dZ - \alpha T(Z,t)\right]$$
$$+ \frac{E}{1-\nu}\left[\frac{1}{2}\int_{-1}^{1} \{\varepsilon''(Z,t)+\Delta\varepsilon''(Z,t)\} dZ - \{\varepsilon''(Z,t)+\Delta\varepsilon''(Z,t)\}\right]$$

The solution can be written in terms of dimensionless quantities by dividing through by the yield stress σ_{yp} and regrouping the other terms as follows:

$$\frac{\sigma(Z,t)}{\sigma_{yp}} = \frac{1}{2}\int_{-1}^{1} \{\frac{E\alpha T(Z,t)}{(1-\nu)\sigma_{yp}}\}dZ - \{\frac{E\alpha T(Z,t)}{(1-\nu)\sigma_{yp}}\}$$

$$+ \frac{1}{2}\int_{-1}^{1} \left[\{\frac{E\varepsilon''(Z,t)}{(1-\nu)\sigma_{yp}}\} + \{\frac{E\Delta\varepsilon''(Z,t)}{(1-\nu)\sigma_{yp}}\}\right]dZ \tag{9}$$

$$- \left[\{\frac{E\varepsilon''(Z,t)}{(1-\nu)\sigma_{yp}}\} + \{\frac{E\Delta\varepsilon''(Z,t)}{(1-\nu)\sigma_{yp}}\}\right]$$

The relationship between the plastic strain and the stress will be shown in the following section in the discussion of the iteration procedure used to solve this equation.

Solution Procedure

Any solution to Equation (9) will satisfy the equilibrium equations, the compatibility equations, and the boundary conditions; however, it is still necessary to satisfy the stress-strain relations of the material. Since no limitations have been placed on the rate of heating or on the frequency of the heating cycle, nonsimultaneous loading and unloading are present in each temperature cycle as, for example, is shown in Figure 5. For this reason proportional loading cannot be assumed to exist and therefore the relationship between stress and plastic strain is not unique. Thus, the necessity of the use of the incremental theory of plasticity which has been previously discussed in Chapter II.

Iteration Procedure

The temperature distribution history is assumed known from a previously performed heat transfer analysis and is divided into a number of finite sized time increments. Equation (9) is solved for the first thermal loading increment by the iteration procedure, i.e., a successive approximation method, of Chapter II as follows:

1. The first approximation of the stress distribution σ is obtained from Equation (9) by assuming that the strains $\Delta\varepsilon''$ and ε'' are zero at each point in the material. The integrals in Equation (9) are evaluated numerically.

2. The equivalent stress distribution σ_e is then computed from Equation (5).

3. The equivalent plastic strain increments $\Delta\varepsilon''_e$ are read from the idealized stress-strain curve of the material at the points where yielding has occurred by the dropping down along a constant strain line to the curve and then reading off the plastic strain only. For the case with the yield stress a function of temperature, a different curve is used for each point for each temperatue change.

4. Values for the plastic strain increments $\Delta\varepsilon''$ are now obtained from Equation (3).

5. Using these values of $\Delta\varepsilon''$ in Equation (9), new approximations are obtained for the stress distribution σ.

6. The procedure is then repeated until the desired accuracy is obtained. The process converges quite rapidly, five or six iterations usually being sufficient. Nevertheless, the question of convergence is discussed more fully in Appendix A.

After a solution is obtained in this manner for the first increment of loading, the values of the plastic strain increments $\Delta\varepsilon''$ are added to the accumulated plastic strain ε'' and the plastic strain increments are again made zero. This procedure is then repeated for the next loading increment of the temperature cycle except that the accumulated plastic strains are carried over from the preceding interval. An example of this stress analysis procedure is illustrated in Appendix B.

This method of analysis will yield very accurate results if infinitesimal time increments and correspondingly infinitesimal changes in the applied thermal loading are considered. However, such a procedure would require an excessive amount of computation; therefore, finite time intervals must be used in a practical analysis. For the computations performed in this study, the time increments are finite, but small enough to obtain reasonably accurate solutions. The selection of the time increment is also discussed in Appendix B.

In order to analyze the resulting structural effects of cyclic thermal loading, it is necessary to carry out this procedure in most cases for three or more cycles resulting in an extremely laborious computational task. The analysis was therefore programmed for the UNIVAC 1108 high speed digital computer. Details of the program are given in Appendix C.

Discussion

Since the present stress analysis of the thermally stressed plate requires the use of numerical and iterative procedures, the relationship between the variables to be investigated and the resulting structural behavior can be acquired only through a very lengthy and indirect manner; that is, the variables must be used in the above stress analysis before the resulting behavior can be determined. Therefore, any simplification which can be made to this analysis, especially if a closed form solution can be obtained, would be highly acceptable if it would result in a clearer understanding or better prediction of the structural behavior of the plate. Thus, an entirely different analysis is developed in the next chapter by expanding upon the "classic" two-bar structural analysis originally developed by Parks [1,2] in explaining the different types of structural behavior.

CHAPTER IV

THEORETICAL ANALYSIS OF TWO-BAR MODEL

General Description

In order to more fully analyze the structural effects of cyclic thermal loading on a free plate, a very simple structural model will be investigated in this chapter. The model consists of two bars fastened together at their ends so that they are constrained to remain of equal lengths (see Figure 6).

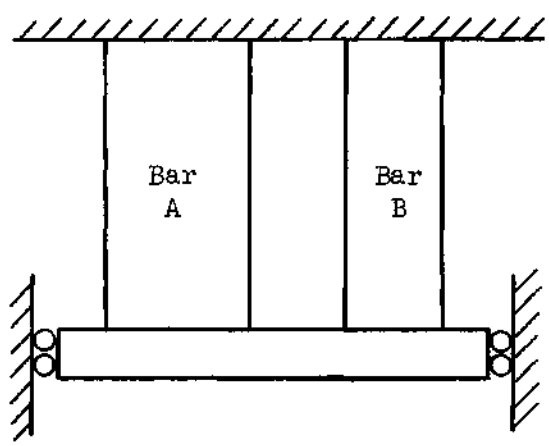


Figure 6. Two-Bar Model

The cross sectional area of bar A is assumed to be larger than the cross sectional area of bar B. Both bars are made of the same material. The material's coefficient of thermal expansion and Young's modulus are assumed constant and its behavior in tension and compression are assumed equal. The material is assumed to exhibit an elastic-perfectly plastic stress-strain relation. The yield stress will be considered constant in one section of the chapter and the additional effect of allowing the yield stress to decrease with temperature will be introduced in a second section.

The temperature cycle experienced by the two bars is one which they would undergo if bar B represented the surface of a structure and bar A the interior region and the temperature of the surrounding atmosphere was cyclic in nature. This temperature cycle is shown in Table 1 where

$$T_I < T_{II}$$

$$\Delta T = T_{II} - T_I$$

$$0 \leq \beta \leq 1$$

and

$$0 \leq \gamma \leq 1$$

Table 1. Temperature Cycle of Two-Bar Model

		a	b	c	d
Bar B	T_I	T_{II}	T_{II}	T_I	T_I
Bar A	$T_I+\beta\Delta T$	$T_I+\beta\Delta T$	$T_I+\gamma\Delta T$	$T_I+\gamma\Delta T$	$T_I+\beta\Delta T$

The cycle is assumed to begin with bar A and bar B at the same temperature, usually $T_I + \beta\Delta T$. Bar B is then heated to T_{II} while bar A remains at the temperature $T_I + \beta\Delta T$. Conduction between the two members then heats bar A up by the amount $(\gamma-\beta)\Delta T$ to the temperature $T_I + \gamma\Delta T$. Then bar B is cooled back down through a ΔT change in temperature to T_I and, to complete the cycle, bar A cools by conduction back down to $T_I + \beta\Delta T$. The letters a, b, c, and d in Table 1 will be used in reference to the various points in the cycle.

Basic Thermoelastic Equations

The elastic equations denoting the change in the stresses in bar A and B when they are subjected to a change in temperature (θ_A and θ_B, respectively) are derived as follows. Let the subscripts A and B denote bars A and B, respectively. From equilibrium of a free body of the bars, the summation of forces in their axial direction yields

$$\Delta\sigma_A A + \Delta\sigma_B B = 0$$

where A and B denotes the cross sectional area of the respective bars. From the stress-strain relations

$$\Delta\varepsilon_A = \frac{\Delta\sigma_A}{E} + \alpha\theta_A$$

$$\Delta\varepsilon_B = \frac{\Delta\sigma_B}{E} + \alpha\theta_B$$

and from geometry, the change in the total strain is the same in both bars, i.e.,

$$\Delta\epsilon_A = \Delta\epsilon_B$$

Solving for the change in the stresses from the above equations yields

$$\Delta\sigma_A = -\frac{E\alpha(\theta_A-\theta_B)R}{1+R}$$

$$\Delta\sigma_B = -\frac{E\alpha(\theta_B-\theta_A)}{1+R}$$

where

$$R = \frac{B}{A}$$

Structural Behavior with Constant Yield Strength

The object of this section is the investigation and development of the modes of behavior of the previously described two-bar structure with the yield stress of the material assumed constant. The following analysis is separated into two parts where different conditions are applied to γ and β. Stress-mechanical strain diagrams will be used to illustrate the different modes of behavior.

Case A. $\gamma+\beta\leq 1$ and Constant Yield Strength

Elastic. Both bars may remain in the elastic region throughout each cycle. If the bars are initially unstressed when their temperatures are equal, then they will remain elastic provided

$$\frac{E\alpha(1-\beta)\Delta T}{1 + R} < \sigma_{yp} \tag{10}$$

which states in essence that bar B does not yield in compression when heated from temperature T_I to temperature T_{II} while bar A remains at temperature $T_I + \beta\Delta T$. This is the point in the cycle where the maximum temperature difference between bar A and bar B occurs; therefore, the maximum stress also occurs at this time. This is illustrated in Figure 7.

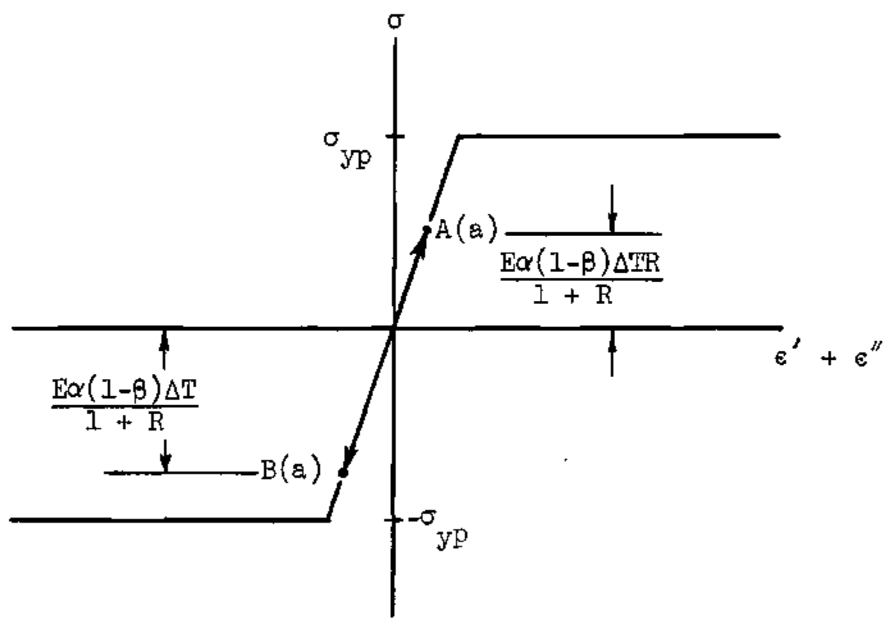


Figure 7. Elastic Behavior of Two-Bar Model in Case A

It should be pointed out that bar A will never yield first since the yield stress of bar A and bar B are the same and are equal in tension and compression, and since the magnitude of the stress in bar B is always greater than that in bar A due to the difference in their cross sectional areas, i.e.,

$$R = \frac{B}{A} < 1$$

Shakedown. As bar B is heated, it will yield in compression during the first temperature cycle before its temperature reaches T_{II} provided

$$\frac{E\alpha(1-\beta)\Delta T}{1 + R} > \sigma_{yp} \tag{11}$$

After the stress in bar B reaches the yield stress $-\sigma_{yp}$, it will yield at a constant stress level until reaching temperature T_{II}. From equilibrium, the tensile stress in bar A will also remain constant during this time. It has already been shown that bar A will never yield before bar B yields and now one sees that, after bar B yields, the stress level in bar A remains constant; therefore, bar A will never yield but will always remain elastic.

The compressive stress in bar B is relieved some when bar A is heated from $T_I + \beta\Delta T$ to $T_I + \gamma\Delta T$ and even more so when its own temperature decreases back to T_I. At this point, bar B will not yield in tension provided

$$\frac{E\alpha(1+\gamma-\beta)\Delta T}{1 + R} < 2\sigma_{yp} \tag{12}$$

This is illustrated in Figure 8.

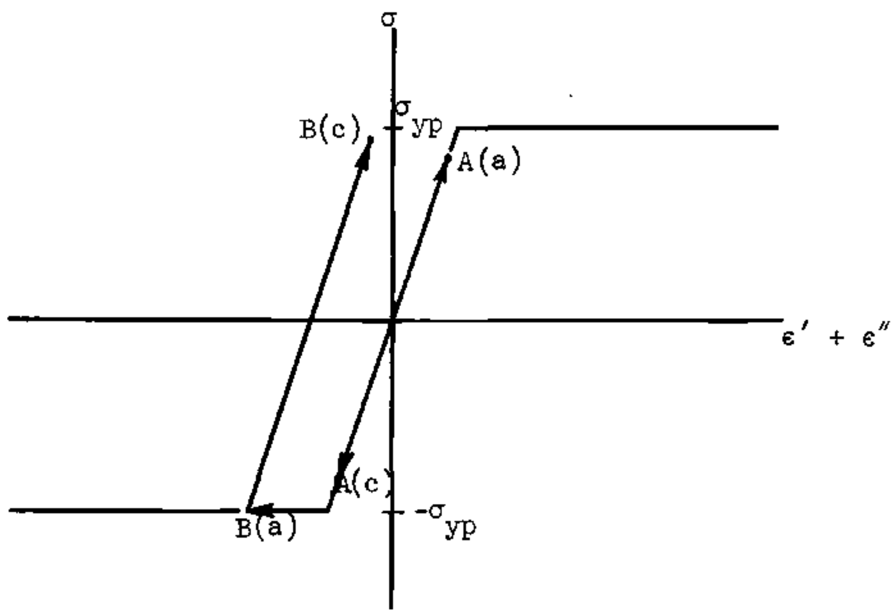


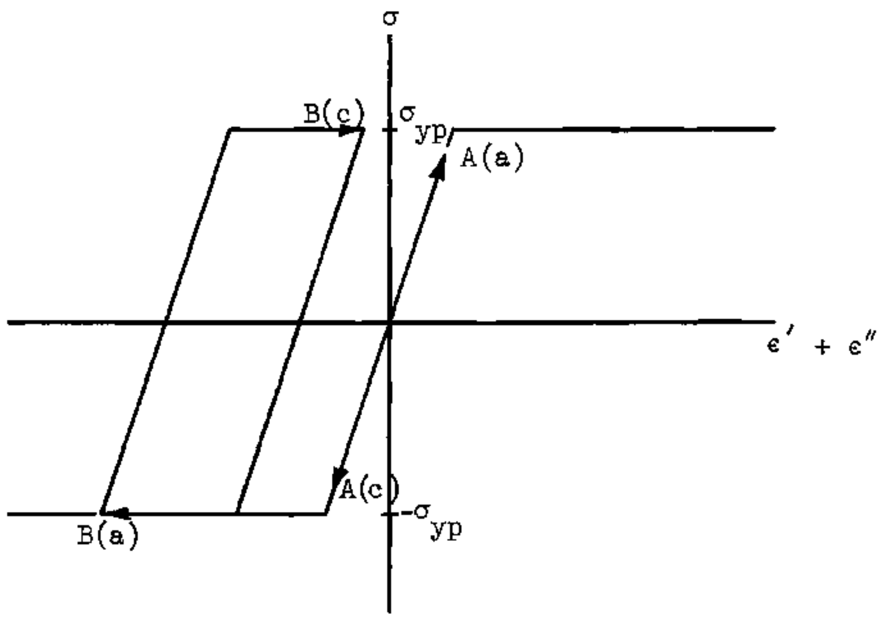
Figure 8. Shakedown Behavior of Two-Bar Model in Case A

In the second and subsequent cycles, the stress in bar B will just reach $-\sigma_{yp}$ at temperature T_{II} producing no further yielding; therefore, the structure has "shaken down" to an elastic state.

<u>Alternate Plasticity</u>. As previously shown, bar B will yield in compression while being heated to T_{II} provided the condition of Equation (11) is satisfied. Upon cooling, bar B will again yield, this time in tension, before its temperature reaches T_I provided

$$\frac{E\alpha(1+\gamma-\beta)\Delta T}{1 + R} > 2\sigma_{yp} \tag{13}$$

This is illustrated in the first part of Figure 9.



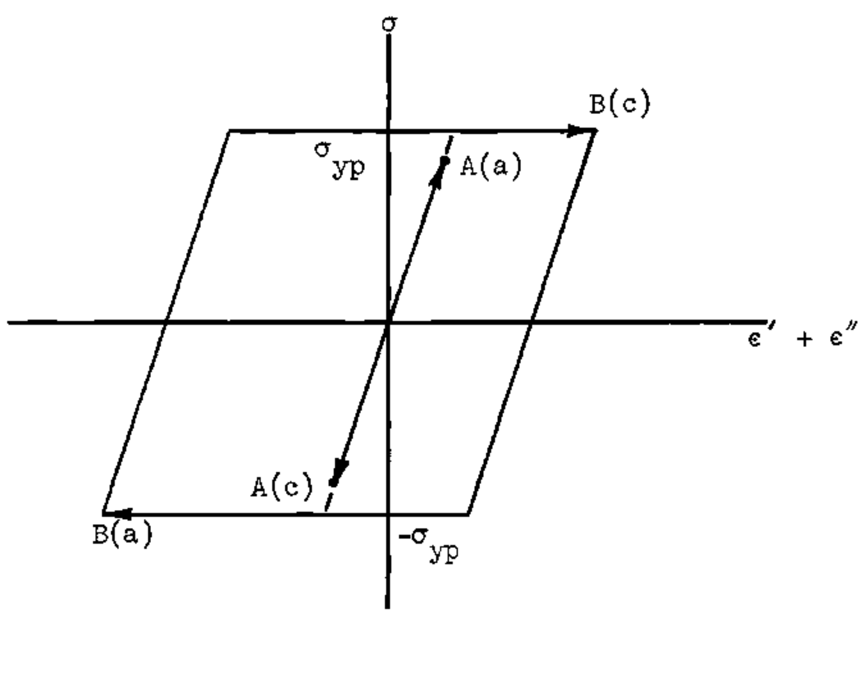


Figure 9. Alternate Plasticity Behavior of Two-Bar Model in Case A

In the second cycle, bar B will again yield in compression as it heats up to T_{II}, provided

$$\sigma_{yp} - \frac{E\alpha(\gamma-\beta)\Delta T}{1 + R} - \frac{E\alpha\Delta T}{1+R} < - \sigma_{yp}$$

which is identical to the condition of Equation (13). Hence, if Equation (13) is satisfied, bar B will experience alternate compressive and tensile yielding, i.e., alternate plasticity, in each temperature cycle while bar A is only subjected to elastic stress levels.

A slightly different reaction may also occur in each temperature cycle. Bar B can begin yielding in tension when bar A is heated to $T_I + \gamma\Delta T$ and then continue yielding in tension as its own temperature is reduced to T_I; in the same manner, it can begin yielding in compression when bar A is cooled to $T_I + \beta\Delta T$ and then continue yielding in compression as its own temperature is again raised to T_{II}, provided that

$$\frac{E\alpha(\gamma-\beta)\Delta T}{1 + R} > 2\sigma_{yp} \tag{14}$$

Under this condition, however, the resulting structural behavior will still be alternate plasticity in bar B and is easily seen to be covered under the condition of Equation (13) above.

The cyclic behavior loop of bar B will contain the origin as shown in the second part of Figure 9 provided that the condition

$$\frac{E\alpha\gamma\Delta T}{1 + R} > \sigma_{yp}$$

is satisfied in addition to the conditions of Equation (11) and Equation (13). The behavior path of bar B in the first cycle depends on whether the cycle begins at the coldest point of the temperature cycle where bar B is at temperature T_I and bar A is at temperature $T_I + \beta\Delta T$ or at the hottest point where bar B is at temperature T_{II} and bar A is at temperature $T_I + \gamma\Delta T$. Yielding begins in bar B in compression for the former and in tension for the latter.

Case B. $\gamma+\beta\geq 1$ and Constant Yield Strength

Bar A will also never yield under this case for the same reasons stated in Case A since the conditions were not dependent upon the values of γ and β. Initially the bars are unstressed when they are both at the same temperature.

Elastic. Bar B will remain elastic throughout each cycle as well as bar A, provided

$$\frac{E\alpha\gamma\Delta T}{1 + R} < \sigma_{yp} \tag{15}$$

This is shown in the stress-mechanical strain diagram of Figure 10. This limiting condition is critical at the point in the temperature cycle where bar B is cooled from temperature T_{II} to temperature T_I while bar A remains at temperature $T_I + \gamma\Delta T$.

Shakedown. This behavior is fundamentally the same as the shakedown behavior under Case A except now yielding occurs at a different

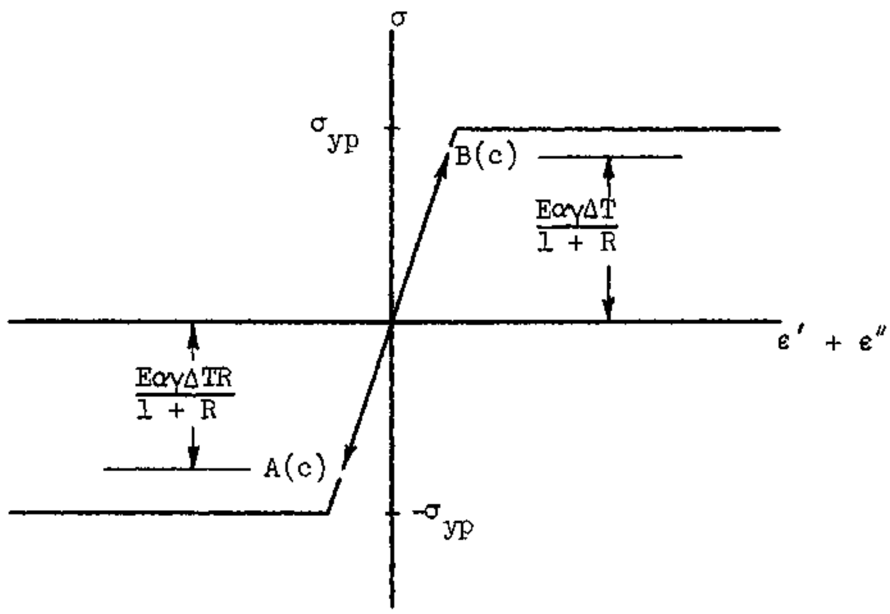


Figure 10. Elastic Behavior of Two-Bar Model in Case B

point in the first temperature cycle. As bar B is cooled in the first temperature cycle, it will yield in tension before its temperature reaches T_I, provided

$$\frac{E\alpha\gamma\Delta T}{1 + R} > \sigma_{yp} \tag{16}$$

After the stress in bar B reaches the yield stress σ_{yp}, it will yield at a constant stress level until its temperature reaches T_I. This tensile stress is relieved some when bar A is cooled from $T_I + \gamma\Delta T$ to $T_I + \beta\Delta T$ and is further relieved when the temperature of bar B is again heated to T_{II} at the start of the second cycle. Bar B will not yield in compression while being heated at this time, provided

$$\frac{E\alpha(1+\gamma-\beta)\Delta T}{1 + R} < 2\sigma_{yp}$$

which is identical to Equation (12). This behavior is illustrated in Figure 11.

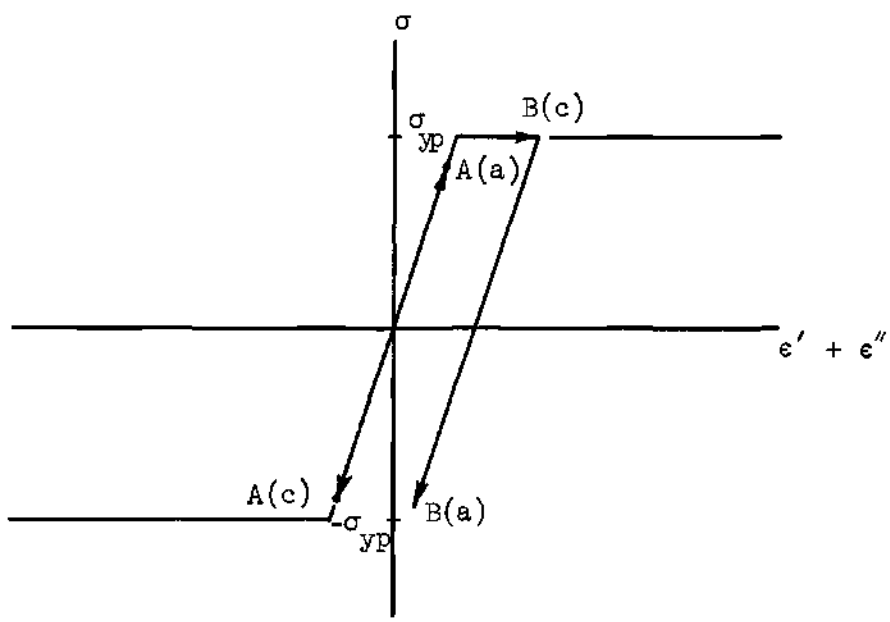


Figure 11. Shakedown Behavior of Two-Bar Model in Case B

Even though bar B yielded in the first cycle while being cooled, it will not yield in the second and subsequent cycles. The stress in bar B will just reach the tensile yield stress σ_{yp} when it is cooled to temperature T_I producing no further yielding; therefore, the structure has "shaken down" to an elastic state.

Alternate Plasticity. This case is practically identical to the alternate plasticity behavior under Case A. For simplicity, the assumption is at first made that the temperature cycle begins at the hottest point of the cycle where bar B is at temperature T_{II} and bar A is at temperature $T_I + \gamma\Delta T$. While being cooled to T_I, bar B will begin yielding in tension, provided that the condition of Equation (16) is satisfied. As bar A cools to $T_I + \beta\Delta T$, the tensile stress in bar B is relieved and will actually reach compressive yielding if the condition of Equation (14) is satisfied. At any rate, bar B will begin or continue yielding in compression while being heated before its temperature reaches T_{II}, provided

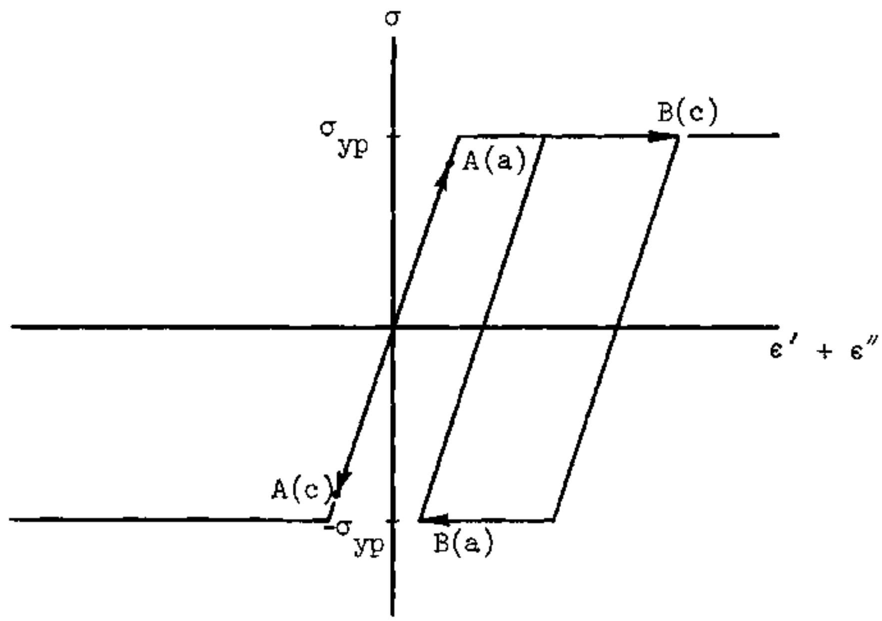
$$\frac{E\alpha(1+\gamma-\beta)\Delta T}{1 + R} > 2\sigma_{yp}$$

which is identical to Equation (13) and covers the condition of Equation (14). This procedure is shown in the first part of Figure 12. If Equation (14) is satisfied, then bar B will again yield in tension as bar A heats up to $T_I + \gamma\Delta T$ and will continue yielding as bar B cools back to T_I. If Equation (14) is not satisfied, then bar B will begin yielding in tension as it cools to T_I, provided

$$-\sigma_{yp} + \frac{E\alpha(\gamma-\beta)\Delta T}{1 + R} + \frac{E\alpha\Delta T}{1+R} > \sigma_{yp}$$

which is the same as Equation (13).

If in addition to the conditions of Equation (13) and Equation (16), the condition of Equation (11) is also satisfied, then the cyclic behavior loops of bar B will contain the origin as shown in the second part of Figure 12. The behavior path of bar B in the first cycle



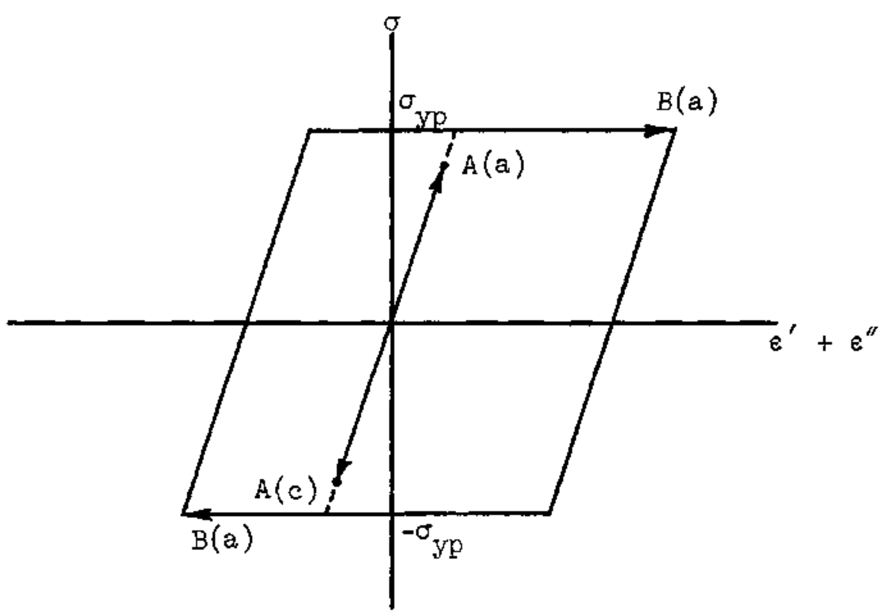


Figure 12. Alternate Plasticity Behavior of Two-Bar Model in Case B

depends on the initial starting point of the cycle. Yielding will begin in bar B in tension if the initial temperatures of bar B and bar A are T_{II} and $T_I + \gamma\Delta T$, respectively, and in compression if the initial temperatures of bar B and bar A are T_I and $T_I + \beta\Delta T$, respectively.

In summary, if Equation (13) is satisfied then bar B will experience alternate plasticity in each temperature cycle. If the condition of Equation (11) as well as that of Equation (13) is present (Case A), then the behavior is similar to that of the first part of Figure 9. But if the condition of Equation (16) as well as that Equation (13) is present (Case B), then the behavior is similar to that of the first part of Figure 12. The behavior loop of bar B will contain the origin in either of these two cases of alternate plasticity if the condition of Equation (16) is also present in Case A and if the condition of Equation (11) is also present in Case B. This is shown in the second part of Figures 9 and 12, respectively.

Discussion of Results

The conditions of Equations (10) through (16) separate the three different types of behavior of the two-bar structure with constant yield strength into zones which can be easily illustrated as functions of two non-dimensional ratios (R, the ratio of the cross sectional area of bar B to that of bar A, and $\frac{E\alpha\Delta T}{\sigma_{yp}}$, the ratio of the range of the thermal strain $\alpha\Delta T$ to the yield strain σ_{yp}/E) and two nondimensional parameters (γ and β, the ratios of the temperature increase of bar A to that of bar B). In this analysis ΔT is a positive quantity by definition since T_{II} is a higher temperature than T_I, and R is assumed to be less than one. Physical constraints also restrict R to be positive since it is the ratio of two areas. Figures 13 and 14 illustrate the two separate cases of $\gamma+\beta\leq 1$ and $\gamma+\beta\geq 1$, respectively (the equality sign holding for both cases).

Structural Behavior with Variation of Yield Strength with Temperature

This section is divided into four parts combining the two conditions which were applied on γ and β in the last section with two additional conditions which deal with the variation of yield stress with temperature. In the last section it was found that bar A would never yield under the condition of a constant and equal yield stress in bar A and B. This condition is of course not found in the present section; therefore, in order to limit this analysis to the same condition of only having bar B yield while bar A remains elastic, the assumption is made that R (area ratio of bar B to bar A) is sufficiently small to force bar A to remain elastic. Since the stress in bar A is directly proportional to R, σ_A can, in this manner, be limited to remain in the elastic zone.

The stress in bar B will vary with the temperature difference between it and bar A at a rate of $-(E\alpha)/(1+R)$ if it remains elastic; therefore, the variation of the yield stress of the material with temperature will be divided into two cases. In one case the slope of the yield stress/temperature curve ($d\sigma_{yp}/dT$) is assumed to be negative but always greater than $-(E\alpha)/(1+R)$, i.e.,

$$0 > \frac{d\sigma_{yp}}{dT} > \frac{-E\alpha}{1 + R}$$

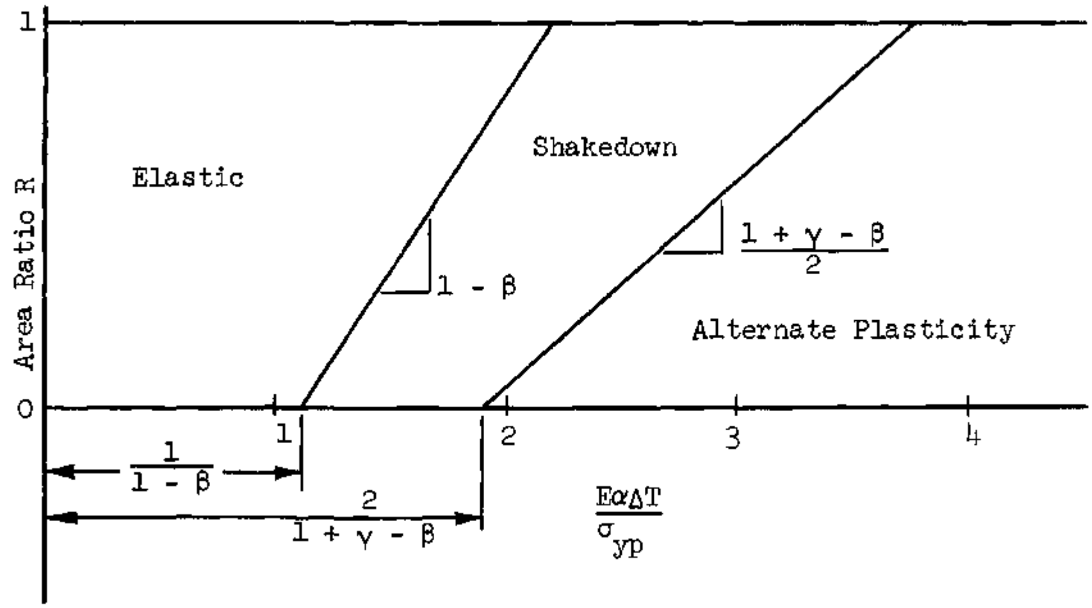


Figure 13. Zones of Behavior of Two-Bar Model in Case A

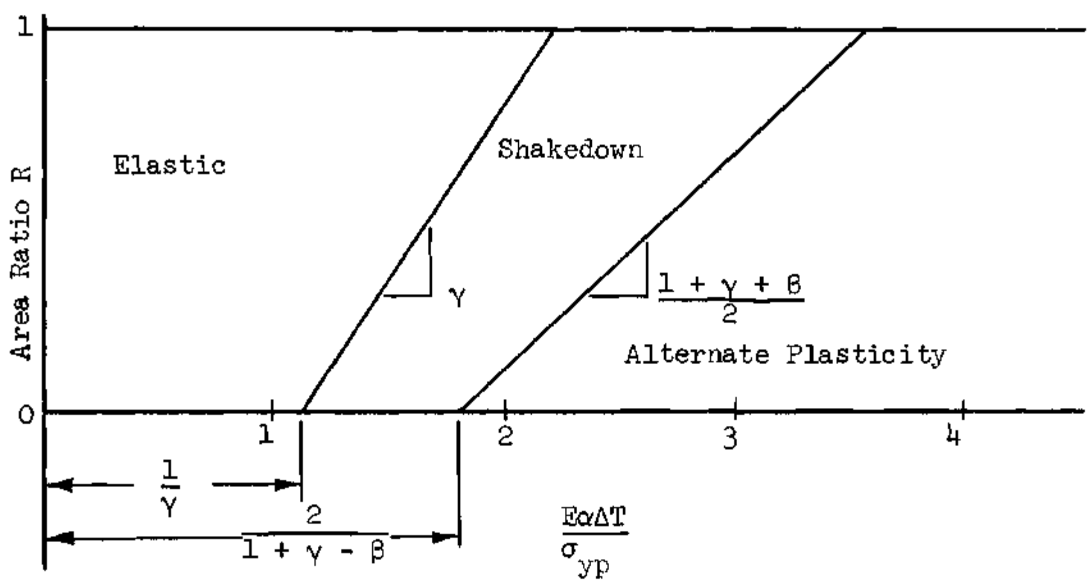


Figure 14. Zones of Behavior of Two-Bar Model in Case B

In the other case the slope is assumed to be less than or equal to $-(E\alpha)/(1+R)$, i.e.,

$$\frac{d\sigma_{yp}}{dT} \leq \frac{-E\alpha}{1 + R}$$

Special cases where the yield stress/temperature curve does not lie completely in one or the other of the above cases for the entire range of temperature from T_I to T_{II} will not be considered here; however, they can be handled on an individual basis in a similar manner.

Since the yield stress, σ_{yp}, is a function of temperature, the yield stress at temperatures T_I, $T_I + \beta\Delta T$, $T_I + \gamma\Delta T$, and T_{II} will be denoted by σ_{T_I}, $\sigma_{\beta T}$, $\sigma_{\gamma T}$, and $\sigma_{T_{II}}$, respectively. Stress-temperature diagrams are conveniently used in this section to illustrate the different modes of behavior instead of stress-mechanical strain diagrams since only repetitive behavior after the first cycle will be considered (incremental collapse would require yielding in bar A as well as in bar B which is not permitted in this section).

Case C

$$\gamma + \beta \leq 1$$

and

$$\frac{d\sigma_{yp}}{dT} \leq \frac{-E\alpha}{1 + R}$$

Elastic. Both bars are initially unstressed when they are at the same temperature, say for instance temperature $T_I + \beta\Delta T$. They will remain elastic as bar B is heated to temperature T_{II} while bar A remains at temperature $T_I + \beta\Delta T$ and their stresses will become

$$\sigma_B = \frac{-E\alpha(1-\beta)\Delta T}{1 + R} \tag{17}$$

$$\sigma_A = \frac{E\alpha(1-\beta)\Delta T R}{1 + R}$$

provided,

$$\frac{E\alpha(1-\beta)\Delta T}{1 + R} < \sigma_{T_{II}} \tag{18}$$

Since the temperature difference between bar A and bar B is a maximum at this point in the temperature cycle, the magnitude of the stress in bar B is a maximum, and since the temperature of bar B itself is a maximum at this point in the temperature cycle, the magnitude of the yield stress of bar B is a minimum; therefore, if yielding does not occur at this time then it cannot possibly occur at any other time in the cycle. This case is illustrated in Figure 15.

Shakedown. As bar B is heated in the first temperature cycle, it will yield in compression before its temperature reaches T_{II}, provided

$$\frac{E\alpha(1-\beta)\Delta T}{1 + R} > \sigma_{T_{II}} \tag{19}$$

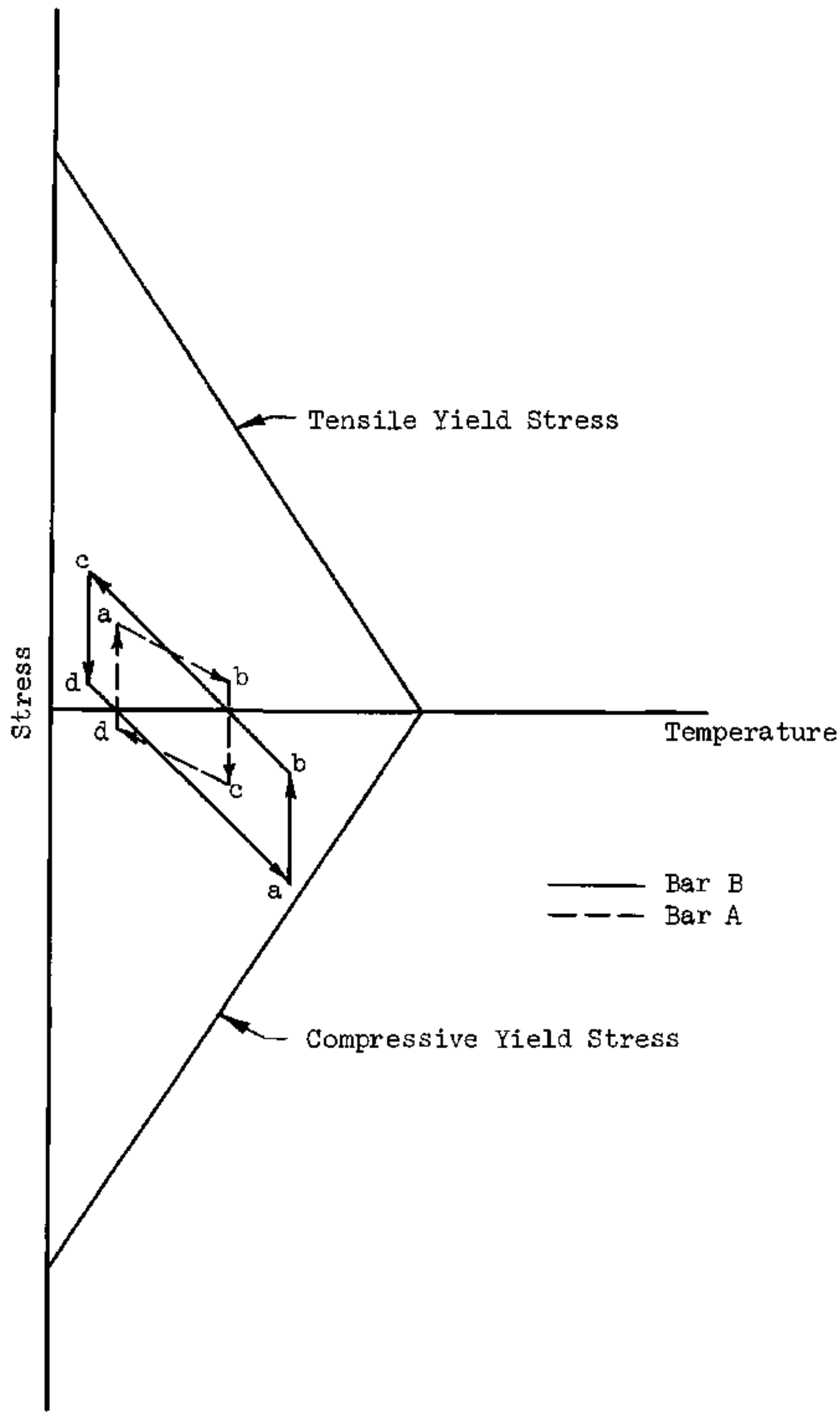


Figure 15. Elastic Behavior of Two-Bar Model in Case C

Hence, the stress in bar B at temperature T_{II} will be $-\sigma_{T_{II}}$. This compressive stress will be relieved some when bar A heats up to temperature $T_I + \gamma\Delta T$ and may even introduce tensile stresses in bar B. Bar B will not yield in tension at this time, provided

$$\frac{E\alpha(\gamma-\beta)\Delta T}{1 + R} < 2\sigma_{T_{II}} \tag{20}$$

As the temperature cycle continues and the temperature of bar B is decreased to T_I, its tensile stress will increase even further. But since its yield stress increases at an even faster rate, as is shown in Figure 16, yielding cannot take place in bar B. Also bar A will not yield at this time, provided

$$\frac{E\alpha(1+\gamma-\beta)\Delta TR}{1 + R} < R\sigma_{T_{II}} + \sigma_{\gamma T} \tag{21}$$

This condition is satisfied for any value of γ, β, or ΔT if R is sufficiently small, in particular if R is less than or equal to δ, where

$$\delta = \frac{\sigma_{\gamma T}}{E\alpha(1+\gamma-\beta)\Delta T}$$

However, δ is not an upper bound for the values of R which satisfy the condition of Equation (21) and should not be considered as such. The stresses are relieved to some degree when bar A is cooled back to temperature $T_I + \beta\Delta T$ to end the first cycle.

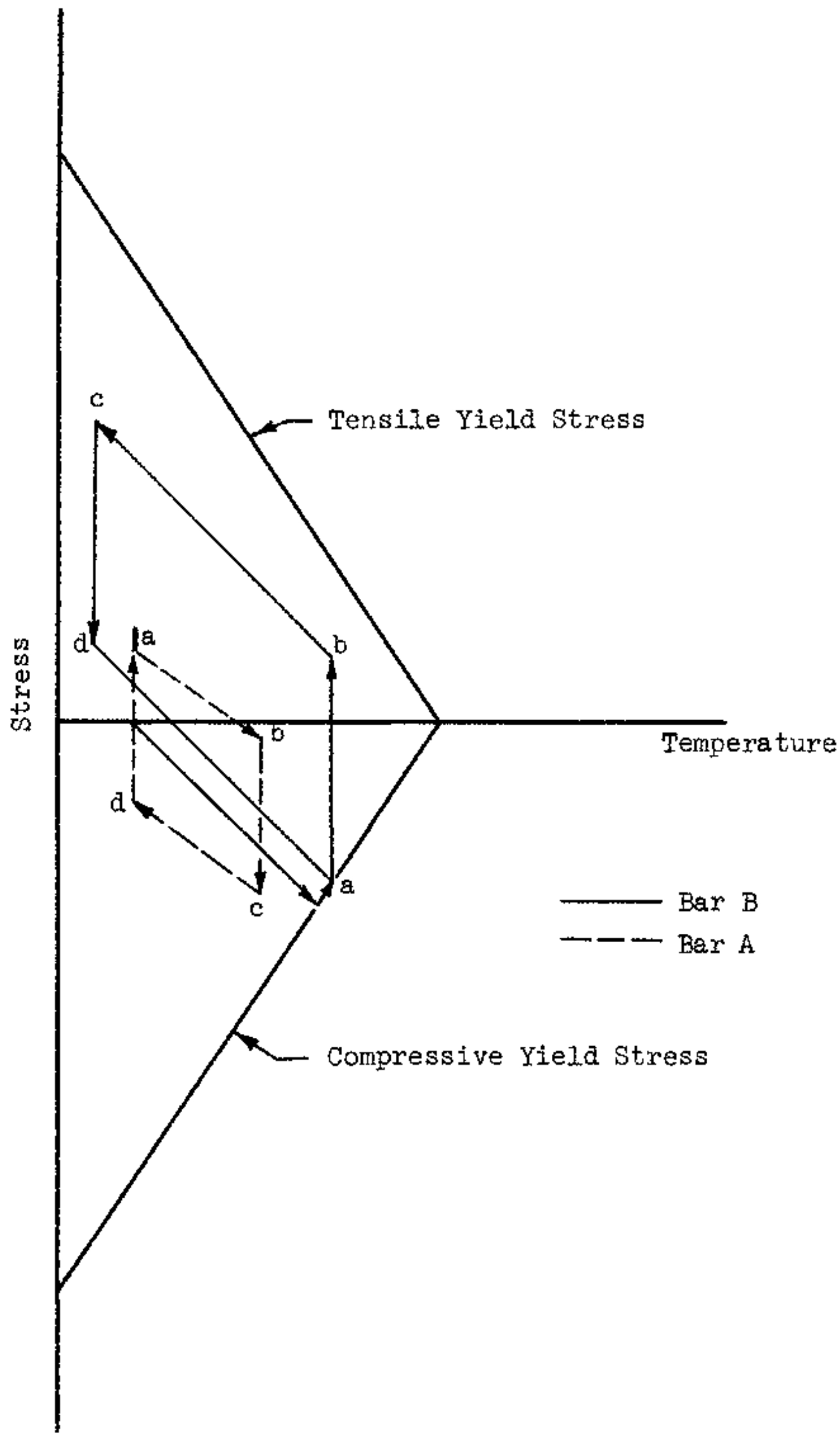


Figure 16. Shakedown Behavior of Two-Bar Model in Case C

In the second and subsequent cycles, the stress in bar B will just reach $-\sigma_{T_{II}}$ at temperature T_{II} resulting in no further inelastic behavior in bar B; therefore, the structure will remain elastic in subsequent cycles and the phenomenon called shakedown has occurred.

Alternate Plasticity. The beginning of this behavior is the same as the beginning of shakedown; that is, as bar B is heated in the first temperature cycle, it will yield in compression before its temperature reaches T_{II} provided the condition of Equation (19) is satisfied. The stress in bar B at temperature T_{II} will thus be $-\sigma_{T_{II}}$. This compressive stress will be relieved as bar A heats up to $T_I + \gamma\Delta T$ and will actually reach tensile yielding if the condition that

$$\frac{E\alpha(\gamma-\beta)\Delta T}{1 + R} > 2\sigma_{T_{II}} \tag{22}$$

is satisfied. As the temperature of bar B is decreased back to T_I, yielding will cease to occur in bar B since its yield stress increases at a faster rate than the induced tensile stress. Bar A will not begin to yield in compression at this time, provided

$$\frac{E\alpha\Delta TR}{1 + R} < \sigma_{\gamma T} - R\sigma_{T_{II}} \tag{23}$$

For a sufficiently small R, this will always be satisfied, that is, if R is less than or equal to δ, where

$$\delta = \frac{\sigma_{\gamma T}}{E\alpha\Delta T + \sigma_{T_{II}}}$$

As before, δ is not an upper bound for the values of R which satisfy the condition of Equation (23).

The stresses are relieved, as shown in Figure 17, when bar A cools back to temperature $T_I + \beta\Delta T$ to end the first cycle. In the second cycle bar B will again yield in compression as it heats up to T_{II} provided

$$\sigma_{T_{II}} + \frac{E\alpha\Delta T}{1+R} - \frac{E\alpha(\gamma-\beta)\Delta T}{1+R} - \frac{E\alpha\Delta T}{1+R} < -\sigma_{T_{II}}$$

which is identical to Equation (22). Therefore, if Equation (22) is satisfied, bar B will experience alternate plasticity in each temperature cycle while bar A always remains elastic, that is, for a sufficiently small R to satisfy the condition of Equation (23).

<u>Case D</u>

$$\gamma + \beta \leq 1$$

and

$$-\frac{E\alpha}{1+R} < \frac{d\sigma_{yp}}{dT} < 0$$

<u>Elastic</u>. The requirement for this case to remain elastic is identical to that of the elastic section of Case C. The cyclic behavior is shown in Figure 18. The satisfying of the condition of Equation (18) will restrict bar B from yielding when it is heated to T_{II}.

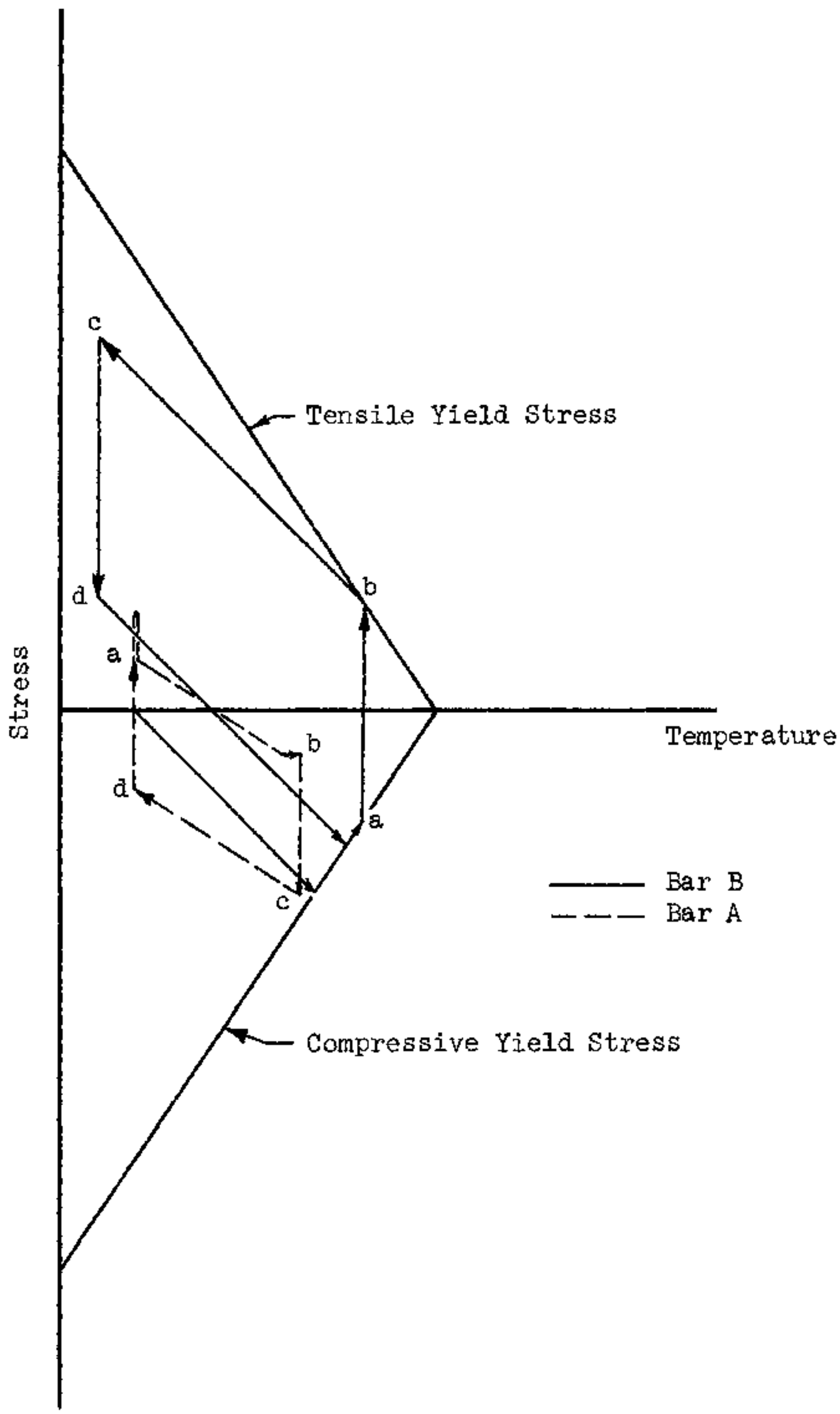


Figure 17. Alternate Plasticity Behavior of Two-Bar Model in Case C

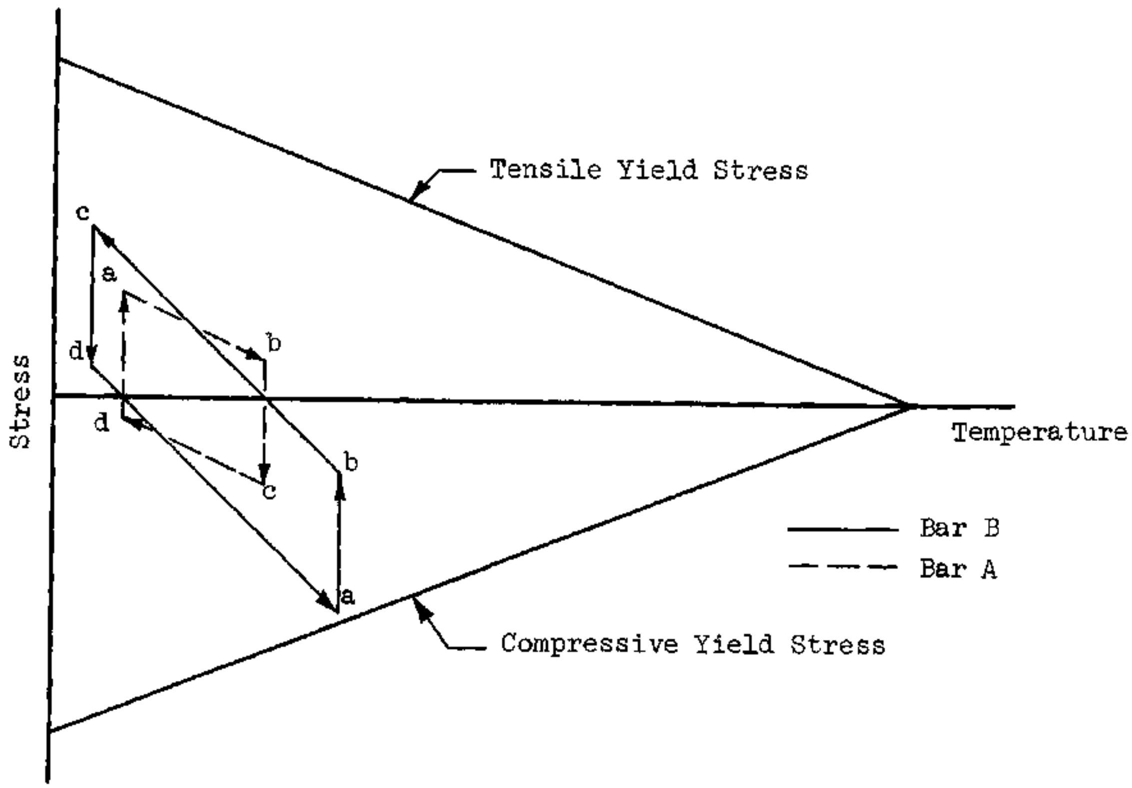


Figure 18. Elastic Behavior of Two-Bar Model in Case D

Shakedown. As bar B heats up to T_{II} in the first cycle, it yields, provided

$$\frac{E\alpha(1-\beta)\Delta T}{1 + R} > \sigma_{T_{II}}$$

which is the same as Equation (19). When the temperature of bar B reaches T_{II}, it will be under a compressive stress of magnitude $\sigma_{T_{II}}$. This compressive stress will be relieved some when bar A is heated to temperature $T_I + \gamma\Delta T$ and may even become a positive tensile stress.

This tensile stress will increase even more as Bar B is cooled back to T_I; nevertheless, bar B will not yield in tension before its temperature reaches T_I, provided

$$\frac{E\alpha(1+\gamma-\beta)\Delta T}{1 + R} < \sigma_{T_I} + \sigma_{T_{II}} \tag{24}$$

Bar A will not yield in compression at this time, provided that the condition on R which was described in the shakedown section of Case C with regard to Equation (21) is met.

The stresses are again reduced some when bar A cools back to temperature $T_I + \beta\Delta T$. In the second and subsequent cycles, the stress in bar B will just reach the yield stress $-\sigma_{T_{II}}$ at temperature T_{II} producing no further yielding in bar B; therefore, the structure will remain elastic in subsequent cycles and the phenomenon called shakedown has occurred. This behavior is illustrated in Figure 19.

Alternate Plasticity. Alternate plasticity begins in the same manner as shakedown with bar B yielding in compression when heated to T_{II} provided Equation (19) is satisfied. After bar A heats up to $T_I + \gamma\Delta T$, bar B will begin cooling and will yield in tension before reaching T_I, provided

$$\frac{E\alpha(1+\gamma-\beta)\Delta T}{1 + R} > \sigma_{T_I} + \sigma_{T_{II}} \tag{25}$$

Bar A will not yield in compression at this time, provided

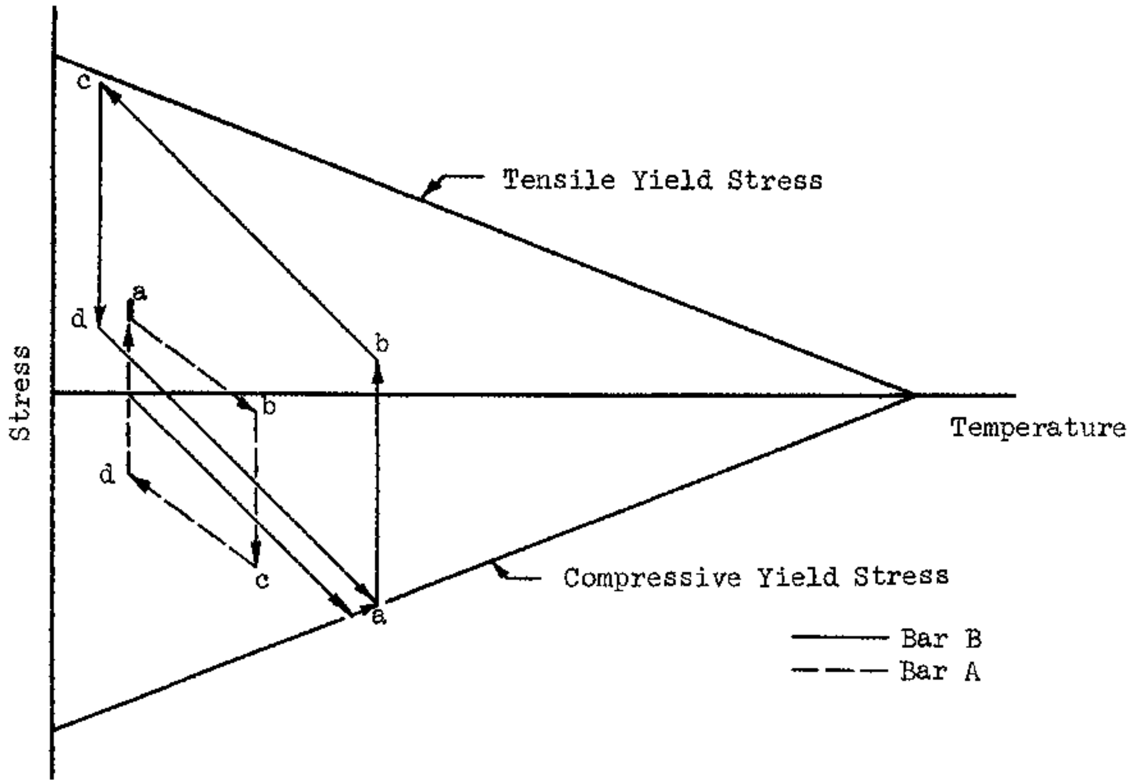


Figure 19. Shakedown Behavior of Two-Bar Model in Case D

$$R < \frac{\sigma_{\gamma T}}{\sigma_{T_I}} \tag{26}$$

The stresses are reduced when bar A cools to its lowest temperature $T_I + \beta\Delta T$ at the end of the first cycle. In the second cycle bar B will again yield in compression as it heats up to T_{II}, provided

$$\sigma_{T_I} - \frac{E\alpha(\gamma-\beta)\Delta T}{1 + R} - \frac{E\alpha\Delta T}{1 + R} < - \sigma_{T_{II}}$$

which is the same as Equation (25). Thus, if the condition of Equation

(25) is satisfied, bar B will experience alternate plasticity in every temperature cycle. Bar A will remain elastic, provided Equation (26) is satisfied. An example of behavior of this type is shown in Figure 20.

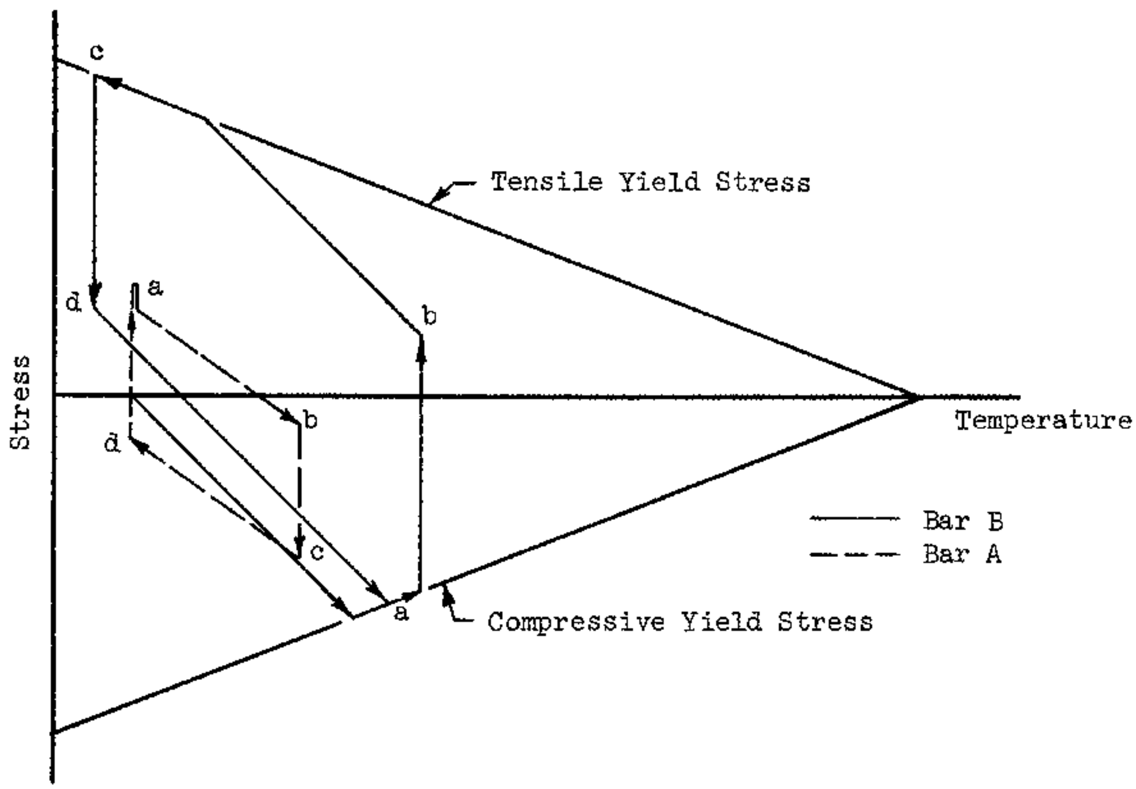


Figure 20. Alternate Plasticity Behavior of Two-Bar Model in Case D

The satisfying of Equation (25) is the minimum requirement to produce alternate plasticity for this case. However, a slightly different behavior from the one shown in Figure 20 will result if

$$\frac{E\alpha(\gamma-\beta)\Delta T}{1 + R} > 2\sigma_{T_{II}}$$

If this condition is satisfied, bar B will begin yielding in tension when bar A is heated to $T_I + \gamma\Delta T$ and then continue yielding as its own temperature is reduced to T_I as shown in Figure 21. A third type of alternate plasticity behavior will result if

$$\frac{E\alpha(\gamma-\beta)\Delta T}{1 + R} > 2\sigma_{T_I}$$

This behavior is quite severe as can be seen in Figure 22. At the end of the first cycle Bar B will begin yielding again in compression as bar A cools to $T_I + \beta\Delta T$ and will continue yielding as it heats up to T_{II}. Then it will begin yielding in tension as bar A is heated to $T_I + \gamma\Delta T$ and will continue yielding as its own temperature reduces back to T_I. Bar A will not yield in either of these cases as long as Equation (26) is satisfied.

Case E

$$\gamma + \beta \geq 1$$

and

$$\frac{d\sigma_{yp}}{dT} \leq \frac{-E\alpha}{1 + R}$$

Elastic. The first requirement for this case to remain elastic is the same as the requirement of the elastic section of Case C and D which is the satisfying of Equation (18). The condition of Equation (18)

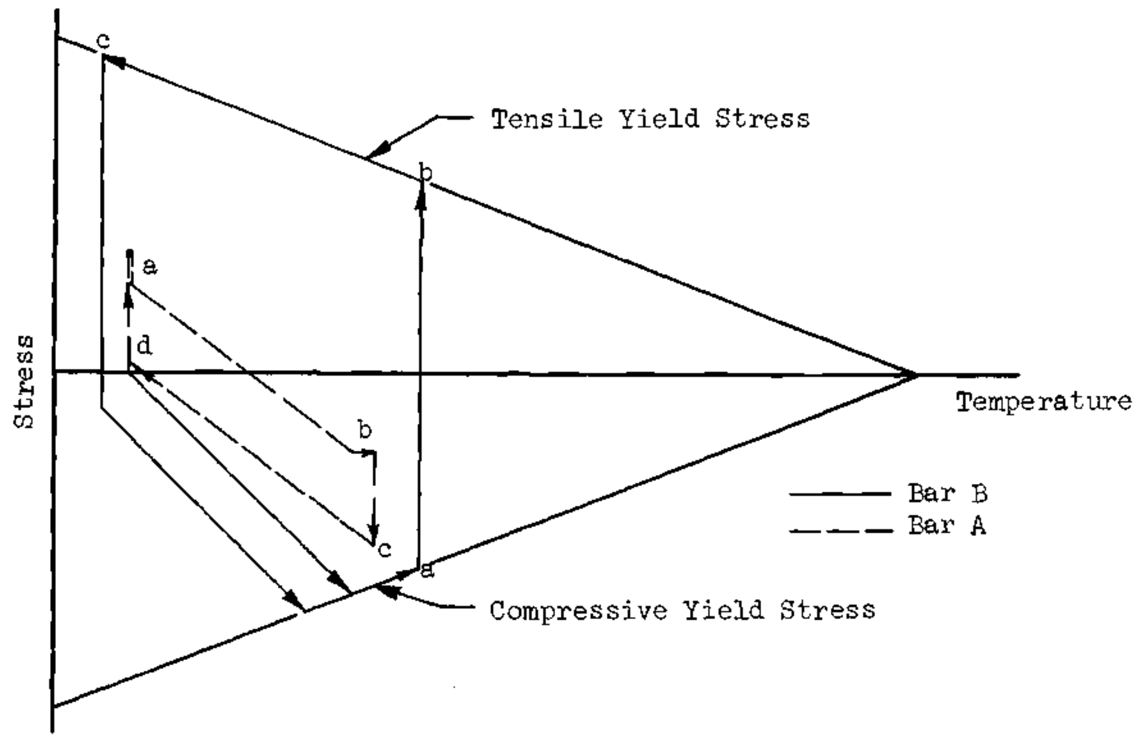


Figure 21. Alternate Plasticity Behavior of Two-Bar Model in Case D

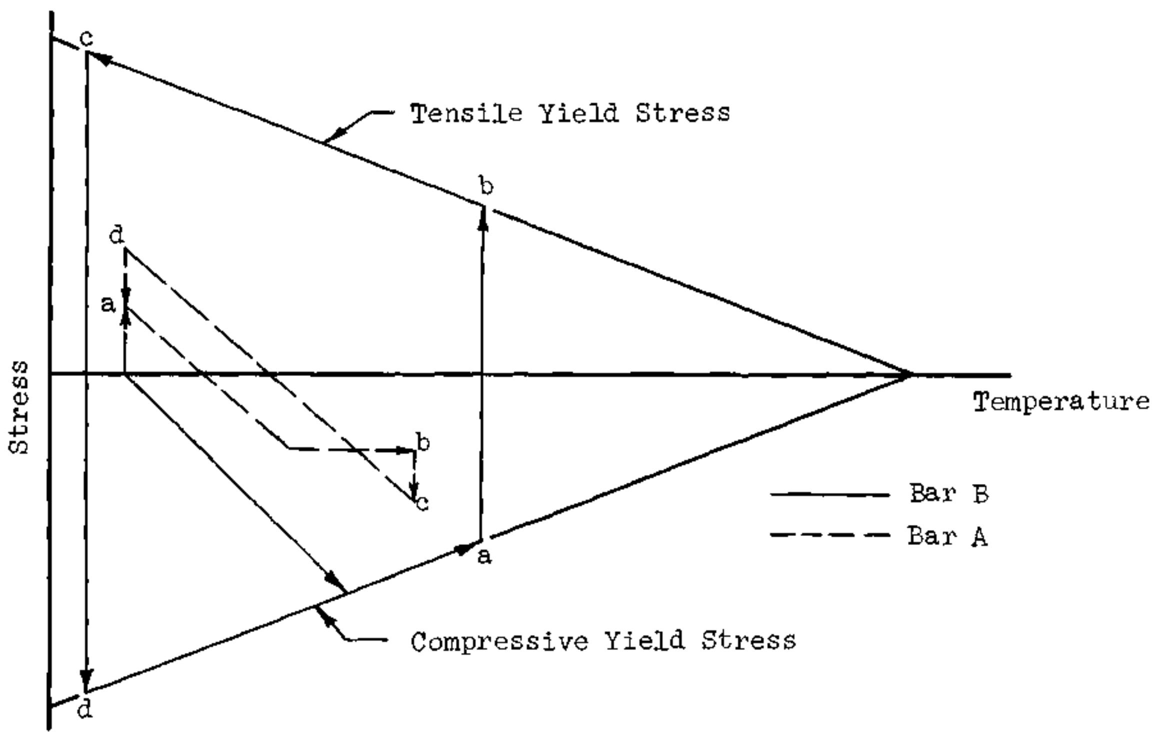


Figure 22. Alternate Plasticity Behavior of Two-Bar Model in Case D

restricts bar B to the elastic region when it is being heated to T_{II}. It was shown for Case C and D that this was the only critical point in the cycle, but for reasons which do not exist in the present case. Here the temperature difference between bar A and bar B is a maximum when bar B is cooled to T_I; therefore, the magnitude of the stress level in bar B is a maximum at point c in Figure 23 instead of at point a. However, since the yield stress of bar B increases even faster than the stress level as it is cooled, yielding cannot take place in bar B at point c.

Bar A will not yield in compression as bar B cools to T_I, provided

$$\frac{E\alpha(\gamma\Delta T)R}{1 + R} < \sigma_{\gamma T}$$

This is satisfied for

$$R < \frac{\sigma_{\gamma T}}{E\alpha\gamma\Delta T - \sigma_{\gamma T}} \qquad (27)$$

Therefore, if Equations (18) and (27) are satisfied, the structure will remain elastic.

Shakedown. The shakedown behavior of this case is identical to that of the shakedown section of Case C. An example of this behavior is shown in Figure 24. The satisfying of Equation (19) will force bar B to yield in compression when it is heated to T_{II} and Equation (20) will keep bar B from yielding in tension when bar A heats up to

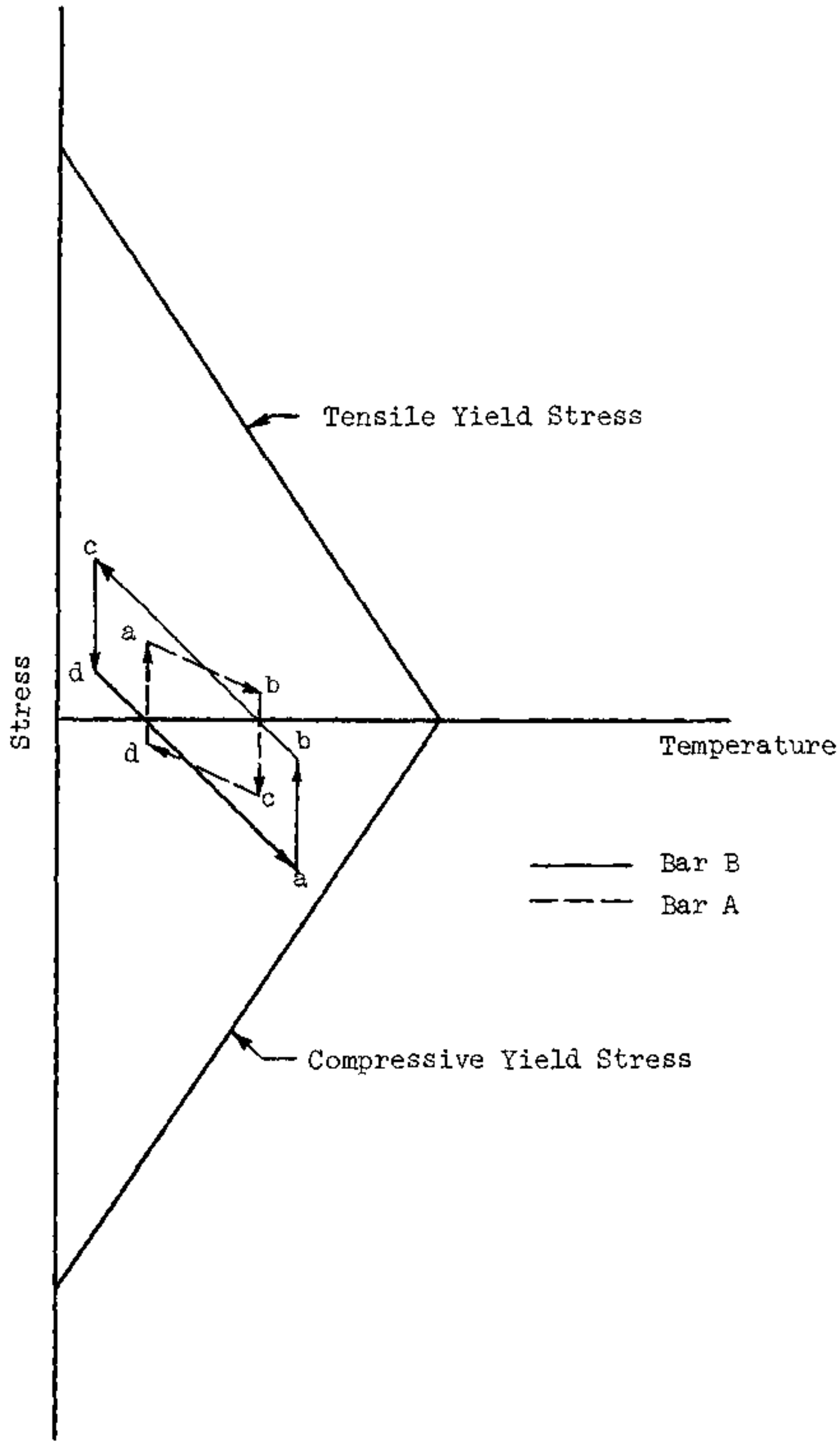


Figure 23. Elastic Behavior of Two-Bar Model in Case E

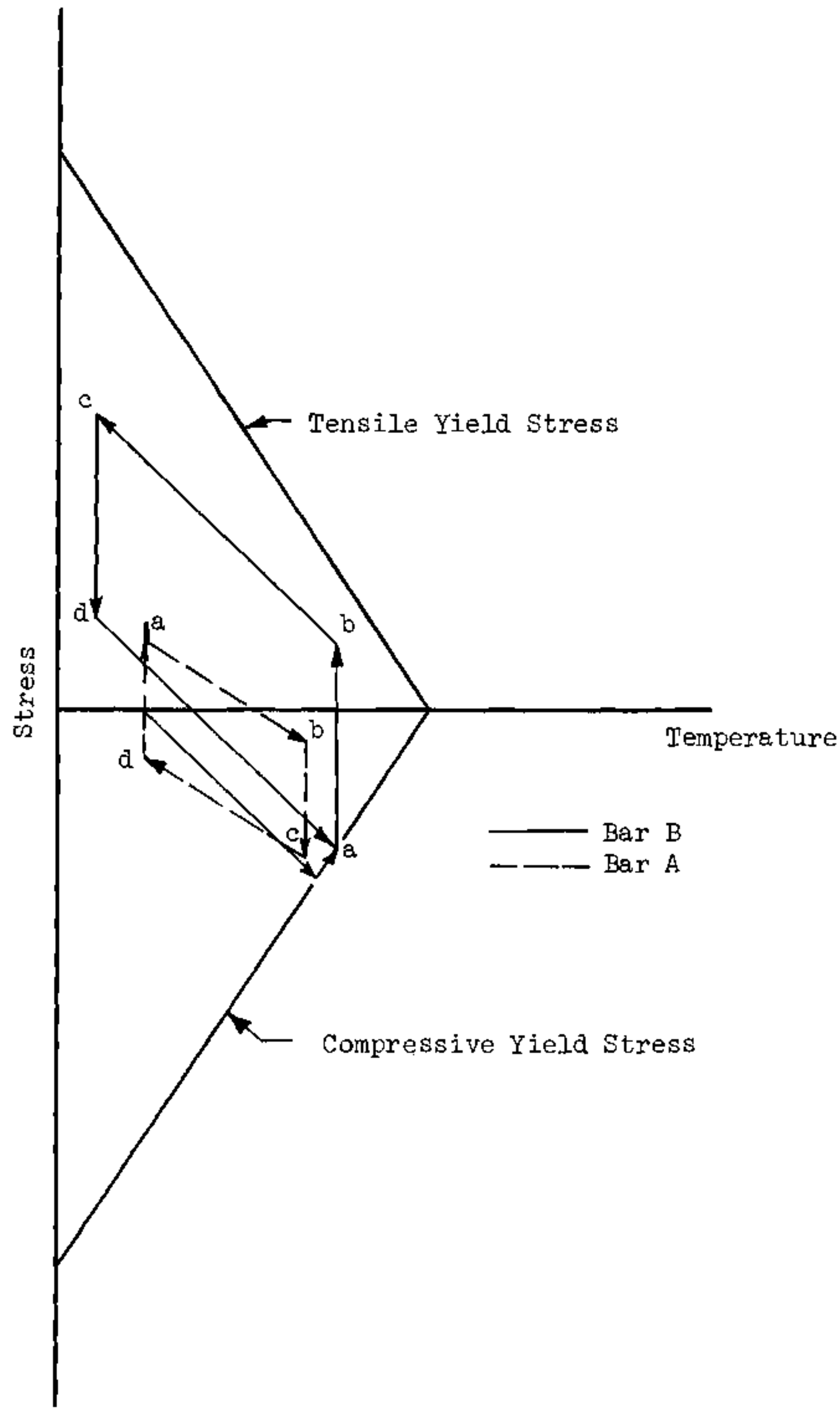


Figure 24. Shakedown Behavior of Two-Bar Model in Case E

$T_I + \gamma\Delta T$. In addition to these two requirements, R must be sufficiently small to satisfy Equation (21) which keeps bar A from yielding when the temperature of bar B cools to T_I.

Alternate Plasticity. The conditions which cause alternate plasticity to occur for this case are the same as those of Case C as can be seen if one compares Figure 25, which illustrates this case, to Figure 17, which illustrates Case C. If Equation (22) is satisfied, bar B will experience alternate plasticity in each temperature cycle while bar A remains elastic, provided R is sufficiently small to satisfy the condition of Equation (23). The difference between Case C and the present case, in the limiting size of the quantity $\gamma + \beta$, has not produced any additional requirements for alternate plasticity to occur.

Case F

$$\gamma + \beta \geq 1$$

and

$$\frac{-E\alpha}{1 + R} < \frac{d\sigma_{yp}}{dT} < 0$$

Elastic. Bar B will remain in the elastic region as it is heated to temperature T_{II}, provided Equation (18), i.e.,

$$\frac{E\alpha(1-\beta)\Delta T}{1 + R} < \sigma_{T_{II}}$$

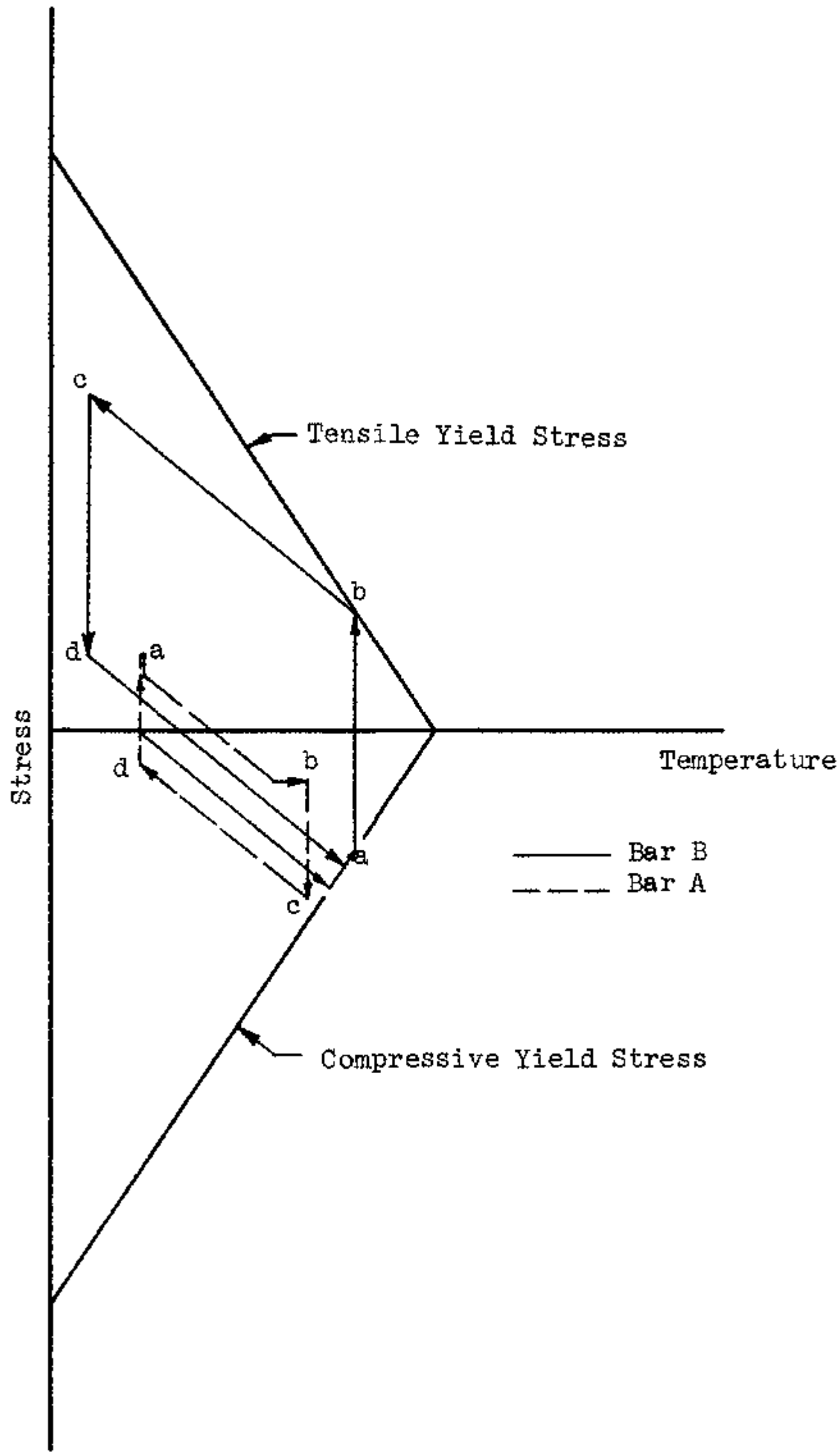


Figure 25. Alternate Plasticity Behavior of Two-Bar Model in Case E

is satisfied. When bar A heats up to $T_I + \gamma\Delta T$, the stresses in the bars will be reduced as shown in Figure 26.

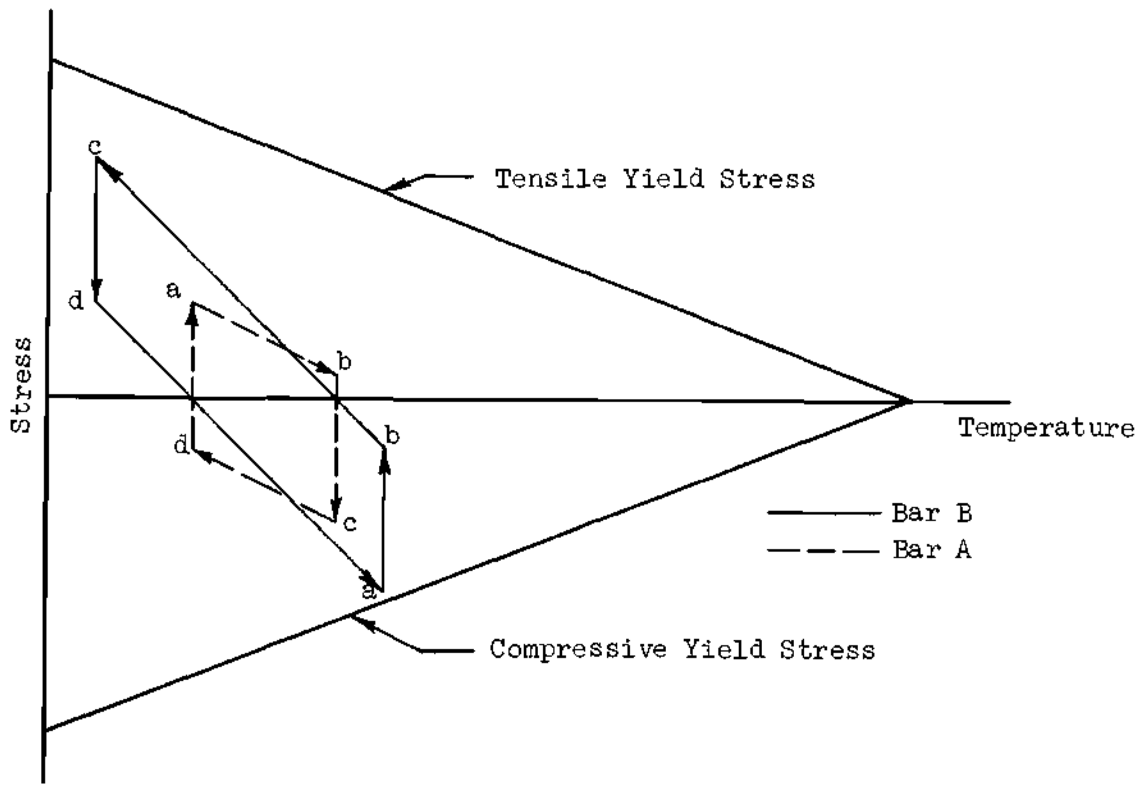


Figure 26. Elastic Behavior of Two-Bar Model in Case F

As the temperature of bar B decreases to T_I its stress will increase but it will not yield in tension, provided

$$\frac{E\alpha\gamma\Delta T}{1 + R} < \sigma_{T_I} \tag{28}$$

Bar A will not yield in compression as bar B cools to T_I if R is sufficiently small to satisfy Equation (27). The stresses are again reduced as bar A cools back to temperature $T_I + \beta\Delta T$ to end the first cycle. Thus, if Equations (18) and (28) are satisfied and, in addition, if R is small enough to satisfy Equation (27), then the bars will remain elastic.

Shakedown. For this case shakedown behavior may result from two different sets of conditions which are discussed separately as follows:

(i) This first set of conditions which can produce shakedown behavior are identical to those in the shakedown section of Case D. The condition of Equation (19) will force bar B to yield when it is heated to temperature T_{II} and Equation (24) will keep it from yielding when it cools back to T_I. Bar A will not yield in compression when bar B cools to T_I, provided R is sufficiently small to satisfy Equation (21) as described in Case C. Shakedown behavior under these conditions are shown in Figure 27.

(ii) The second set of conditions which can produce shakedown behavior are as follows. If Equation (19) is not satisfied, bar B will not yield when it is heated to T_{II}. However, as it is cooled from T_{II} to T_I it will yield in tension, provided

$$\frac{E\alpha\gamma\Delta T}{1 + R} > \sigma_{T_I} \tag{29}$$

When the temperature of bar B reaches T_I, its stress will be σ_{T_I}. Bar A will not yield in compression as bar B is cooled, provided Equation (27), i.e.,

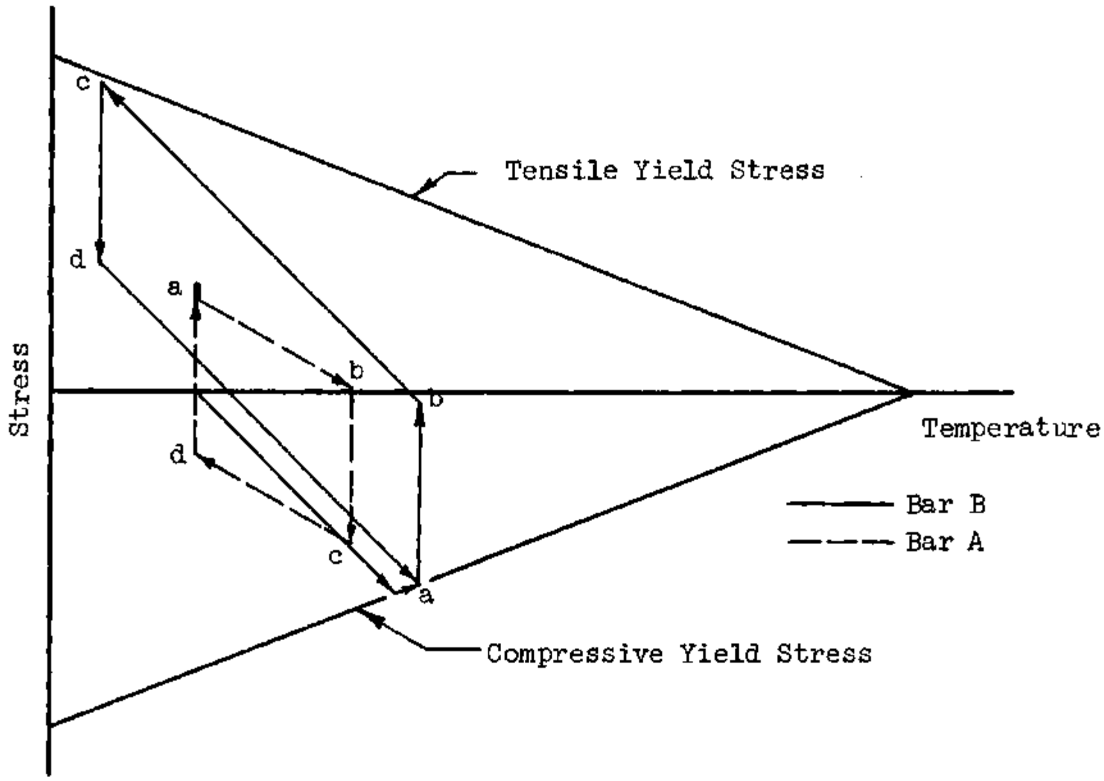


Figure 27. Shakedown Behavior (i) of Two-Bar Model in Case F

$$R < \frac{\sigma_{\gamma T}}{E\alpha\gamma\Delta T - \sigma_{\gamma T}}$$

is satisfied. The tensile stress in bar B will be relieved when bar A cools to temperature $T_I + \beta\Delta T$ and may even become compressive. As bar B is heated again to T_{II}, it will not yield in compression, provided Equation (24), i.e.,

$$\frac{E\alpha(1+\gamma-\beta)\Delta T}{1 + R} < \sigma_{T_I} + \sigma_{T_{II}}$$

is satisfied. The stresses are again reduced when bar A is heated to $T_I + \gamma\Delta T$. When bar B is cooled back to T_I, its stress will just reach the tensile yield stress σ_{T_I} producing no further yielding in bar B. The structure will therefore remain elastic in subsequent cycles as illustrated in Figure 28.

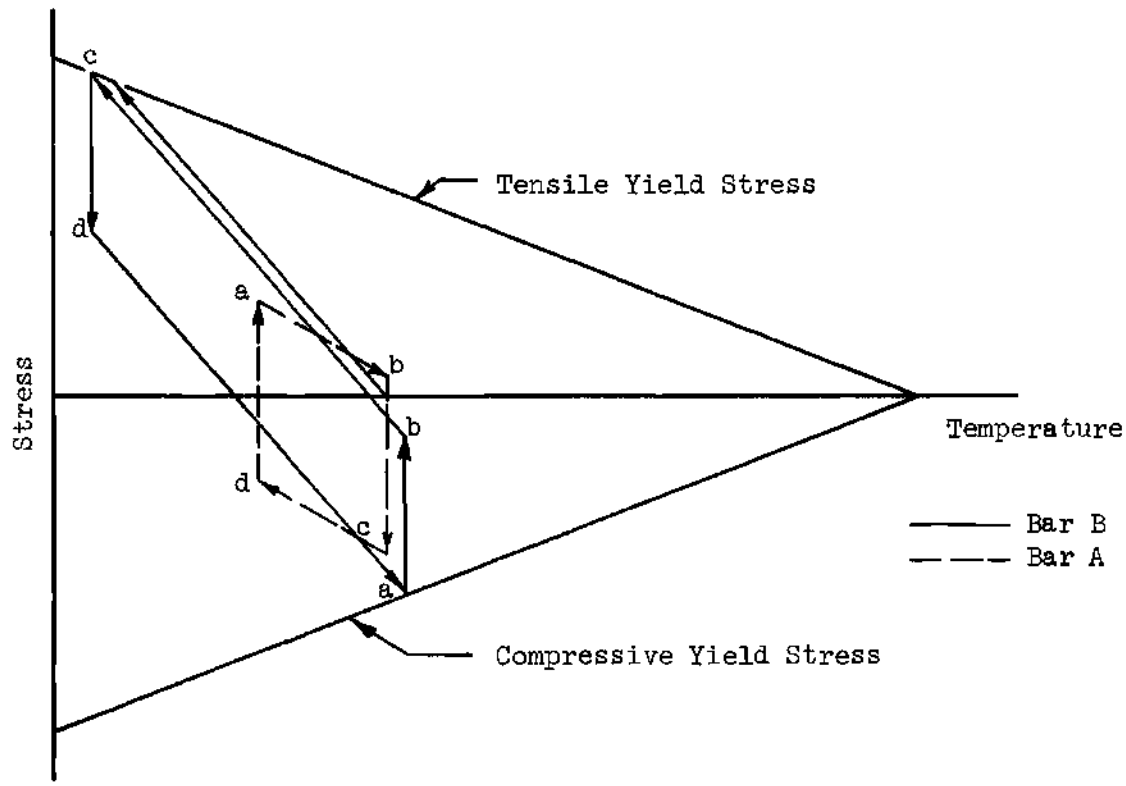


Figure 28. Shakedown Behavior (ii) of Two-Bar Model in Case F

Alternate Plasticity. Alternate plasticity for this case is identical to that of Case D. As shown in that case, the minimum requirement to produce alternate plasticity is for Equations (25) and (26) to be satisfied. Two slightly different types of alternate plasticity behavior were also discussed in Case D and the equations which would have to be satisfied for each to occur were also given. Graphs showing these three behaviors for this case are given in Figures 29, 30, and 31.

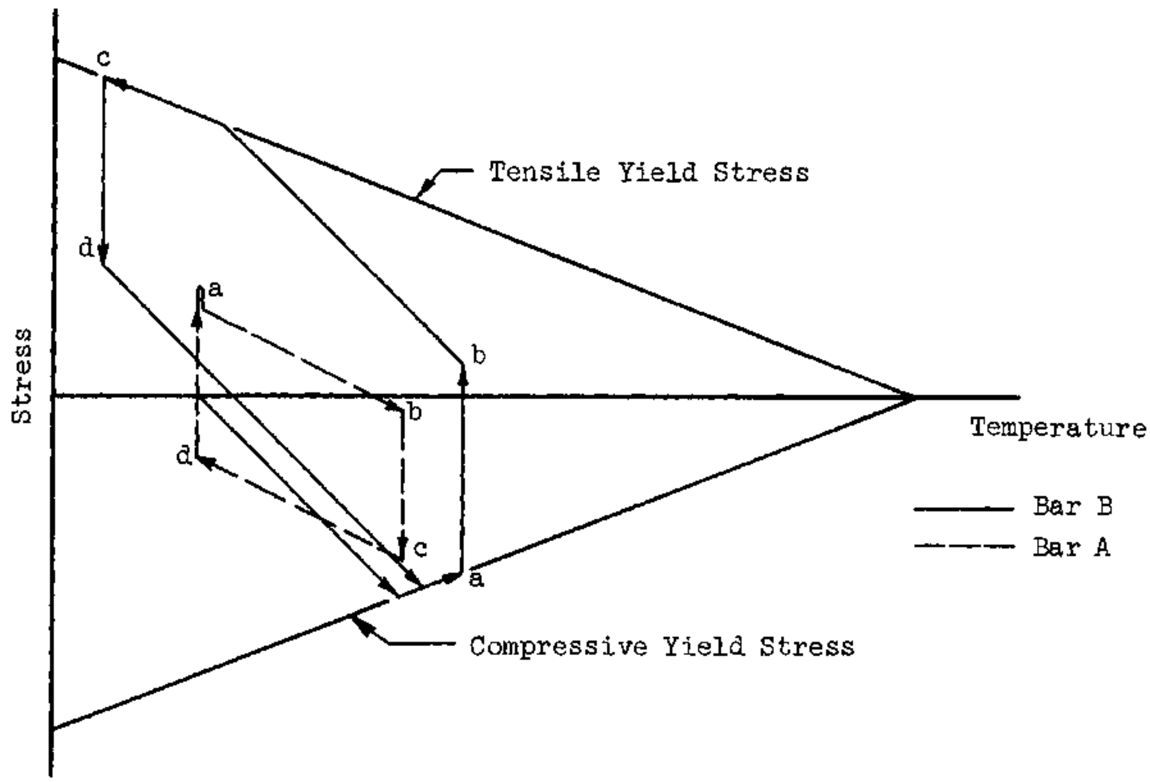


Figure 29. Alternate Plasticity Behavior of Two-Bar Model in Case F

Discussion of Results

Equations (18) through (29) specify conditions for the occurrence of the three types of behavior, elastic, shakedown and alternate plasticity, in four separate cases which depend upon the relative values of γ and β and upon the slope of the yield stress/temperature curve.

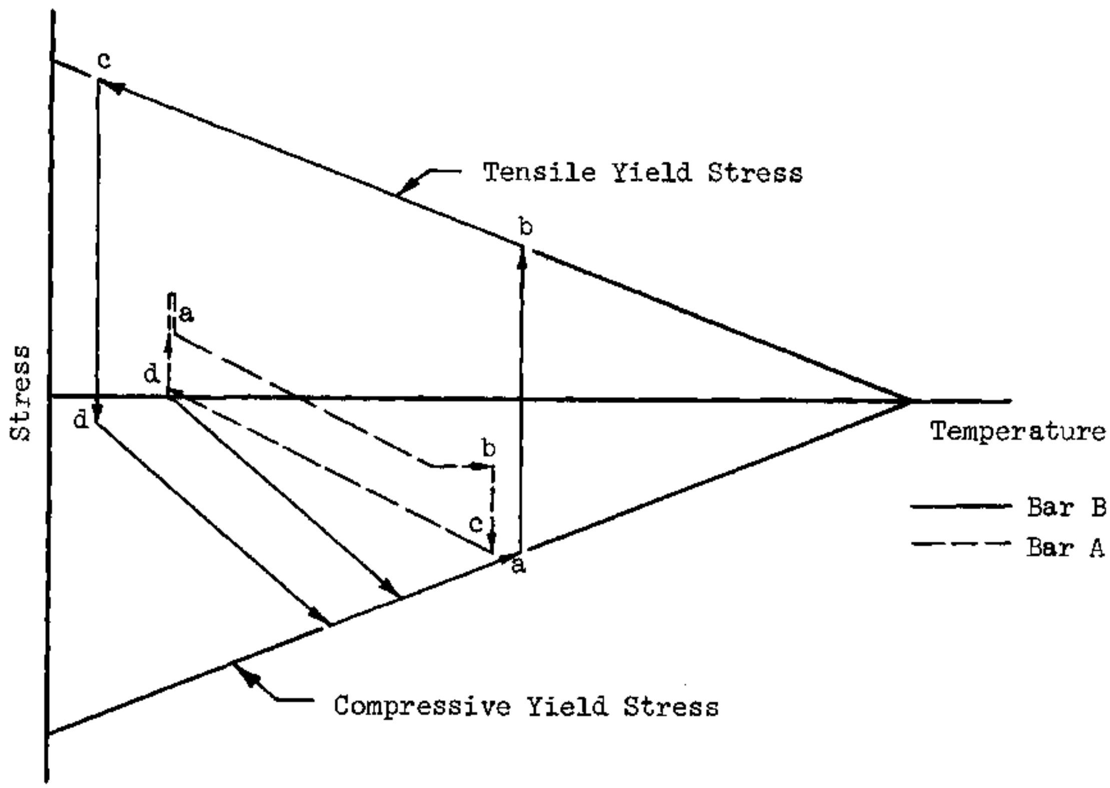


Figure 30. Alternate Plasticity Behavior of Two-Bar Model in Case F

These equations are presented for convenience in Table 2.

So far the only condition constraining the form of the yield stress/temperature curve is that the yield stress should decrease with temperature. The analysis will now be restricted for convenience to a linear yield stress-temperature relationship; however, the method of analysis can be easily extended to more general curves. If the yield stress/temperature curve is a straight line of slope λ, then

$$\frac{d\sigma_{yp}}{dT} = \lambda$$

Table 2. Conditions Defining the Modes of Behavior of the Two-Bar Model in Cases C, D, E, and F

	Case C $\left\{\begin{array}{c}\gamma+\beta\le 1\\ \frac{d\sigma_{yp}}{dT}\le\frac{-E\alpha}{1+R}\end{array}\right\}$	Case D $\left\{\begin{array}{c}\gamma+\beta\le 1\\ \frac{-E\alpha}{1+R}<\frac{d\sigma_{yp}}{dT}<0\end{array}\right\}$	Case E $\left\{\begin{array}{c}\gamma+\beta\ge 1\\ \frac{d\sigma_{yp}}{dT}\le\frac{-E\alpha}{1+R}\end{array}\right\}$	Case F $\left\{\begin{array}{c}\gamma+\beta\ge 1\\ \frac{-E\alpha}{1+R}<\frac{d\sigma_{yp}}{dT}<0\end{array}\right\}$
ELASTIC	(18) $\frac{E\alpha(1-\beta)\Delta T}{1+R}<\sigma_{T_{II}}$	(18) $\frac{E\alpha(1-\beta)\Delta T}{1+R}<\sigma_{T_{II}}$	(18) $\frac{E\alpha(1-\beta)\Delta T}{1+R}<\sigma_{T_{II}}$ (27) $R<\frac{\sigma_{\gamma T}}{E\alpha\gamma\Delta T-\sigma_{\gamma T}}$	(18) $\frac{E\alpha(1-\beta)\Delta T}{1+R}<\sigma_{T_{II}}$ (28) $\frac{E\alpha\gamma\Delta T}{1+R}<\sigma_{T_I}$ (27) $R<\frac{\sigma_{\gamma T}}{E\alpha\gamma\Delta T-\sigma_{\gamma T}}$
SHAKEDOWN	(19) $\frac{E\alpha(1-\beta)\Delta T}{1+R}>\sigma_{T_{II}}$ (20) $\frac{E\alpha(\gamma-\beta)\Delta T}{1+R}<2\,\sigma_{T_{II}}$ (21) $\frac{E\alpha(1+\gamma-\beta)\Delta TR}{1+R}<R\sigma_{T_{II}}+\sigma_{\gamma T}$	(19) $\frac{E\alpha(1-\beta)\Delta T}{1+R}>\sigma_{T_{II}}$ (24) $\frac{E\alpha(1+\gamma-\beta)\Delta T}{1+R}<\sigma_{T_I}+\sigma_{T_{II}}$ (21) $\frac{E\alpha(1+\gamma-\beta)\Delta TR}{1+R}<R\sigma_{T_{II}}+\sigma_{\gamma T}$	(19) $\frac{E\alpha(1-\beta)\Delta T}{1+R}>\sigma_{T_{II}}$ (20) $\frac{E\alpha(\gamma-\beta)\Delta T}{1+R}<2\,\sigma_{T_{II}}$ (21) $\frac{E\alpha(1+\gamma-\beta)\Delta TR}{1+R}<R\sigma_{T_{II}}+\sigma_{\gamma T}$	(19) $\frac{E\alpha(1-\beta)\Delta T}{1+R}>\sigma_{T_{II}}$ (24) $\frac{E\alpha(1+\gamma-\beta)\Delta T}{1+R}<\sigma_{T_I}+\sigma_{T_{II}}$ (21) $\frac{E\alpha(1+\gamma-\beta)\Delta TR}{1+R}<R\sigma_{T_{II}}+\sigma_{\gamma T}$ (24) $\frac{E\alpha(1+\gamma-\beta)\Delta T}{1+R}<\sigma_{T_I}+\sigma_{T_{II}}$ (29) $\frac{E\alpha\gamma\Delta T}{1+R}>\sigma_{T_I}$ (27) $R<\frac{\sigma_{\gamma T}}{E\alpha\gamma\Delta T-\sigma_{\gamma T}}$
ALTERNATE PLASTICITY	(22) $\frac{E\alpha(\gamma-\beta)\Delta T}{1+R}>2\,\sigma_{T_{II}}$ (23) $\frac{E\alpha\Delta TR}{1+R}<\sigma_{\gamma T}-R\sigma_{T_{II}}$	(25) $\frac{E\alpha(1+\gamma-\beta)\Delta T}{1+R}>\sigma_{T_I}+\sigma_{T_{II}}$ (26) $R<\frac{\sigma_{\gamma T}}{\sigma_{T_I}}$	(22) $\frac{E\alpha(\gamma-\beta)\Delta T}{1+R}>2\,\sigma_{T_{II}}$ (23) $\frac{E\alpha\Delta TR}{1+R}<\sigma_{\gamma T}-R\sigma_{T_{II}}$	(25) $\frac{E\alpha(1+\gamma-\beta)\Delta T}{1+R}>\sigma_{T_I}+\sigma_{T_{II}}$ (26) $R<\frac{\sigma_{\gamma T}}{\sigma_{T_I}}$

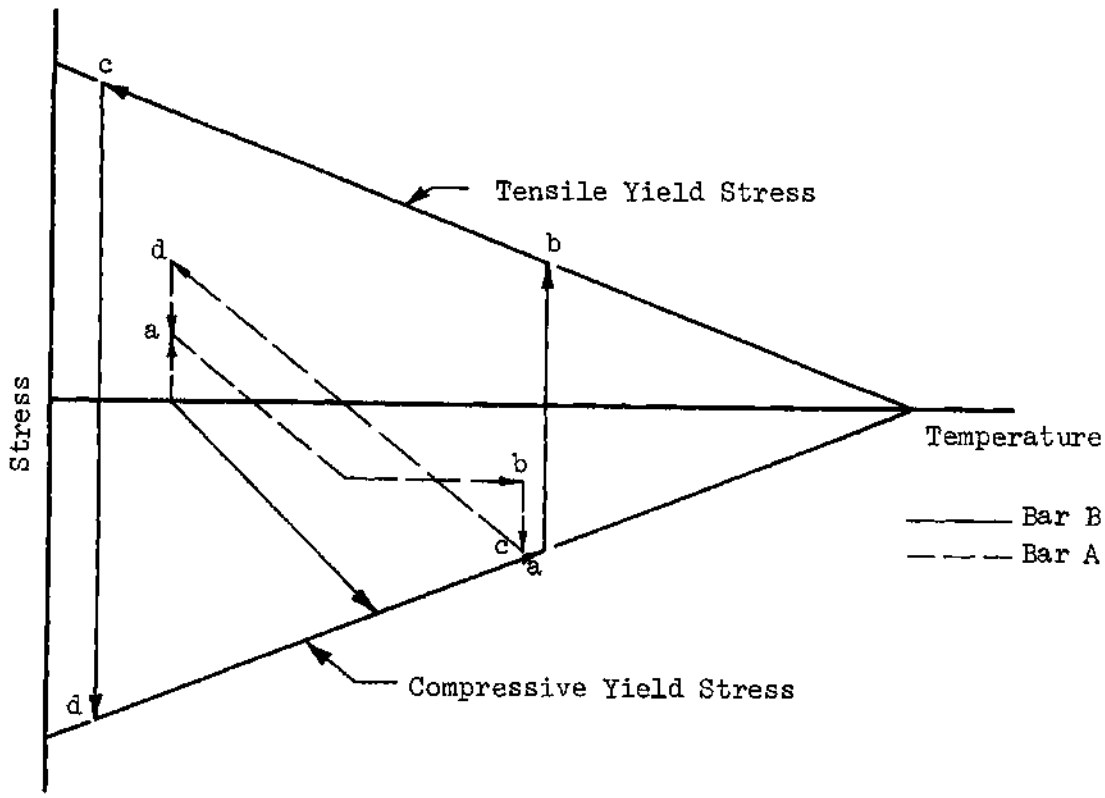


Figure 31. Alternate Plasticity Behavior of Two-Bar Model in Case F

and

$$\sigma_{T_{II}} = \sigma_{T_I} + \lambda \Delta T \tag{30}$$

Since this analysis is restricted to negative values of λ, ΔT has an upper limit at which the strength of the material becomes zero,

$$\Delta T_{(max)} = - \frac{\sigma_{T_I}}{\lambda}$$

The maximum allowable temperature therefore is

$$T_{(max)} = T_I + \Delta T_{(max)}$$

or

$$T_{(max)} = T_I - \frac{\sigma_{T_I}}{\lambda}$$

Substitution of Equation (30) into the equations presented in Table 2 defining the boundaries between the zones of behavior for Cases C, D, E, and F results in

$$\frac{E\alpha(1-\beta)\Delta T}{1 + R} = \sigma_{T_{II}}$$

becoming

$$\frac{E\alpha\Delta T}{\sigma_{T_I}} = \frac{1 + R}{(1-\beta) - \frac{\lambda(1+R)}{E\alpha}} \tag{31}$$

and

$$\frac{E\alpha(\gamma-\beta)\Delta T}{1 + R} = 2\sigma_{T_{II}}$$

becoming

$$\frac{E\alpha\Delta T}{\sigma_{T_I}} = \frac{1 + R}{\frac{(\gamma-\beta)}{2} - \frac{\lambda(1+R)}{E\alpha}} \tag{32}$$

and

$$\frac{E\alpha(1+\gamma-\beta)\Delta T}{1 + R} - \sigma_{T_I} + \sigma_{T_{II}}$$

becoming

$$\frac{E\alpha\Delta T}{\sigma_{T_I}} = \frac{2(1+R)}{(1+\gamma-\beta) - \frac{\lambda(1+R)}{E\alpha}} \tag{33}$$

By making the corresponding substitution of Equations (31), (32), and (33) into Table 2, the results of this analysis for a linear yield stress-temperature relationship is thereby obtained and is presented in Table 3.

Table 3. Conditions Defining the Modes of Behavior of the Two-Bar Model in Cases C, D, E, and F for a Linear Yield Stress-Temperature Relationship

	Case C $\left\{ \begin{array}{c} \gamma + \beta \le 1 \\ \frac{d\sigma_{yp}}{dT} \le \frac{-E\alpha}{1+R} \end{array} \right\}$	Case D $\left\{ \begin{array}{c} \gamma + \beta \le 1 \\ \frac{-E\alpha}{1+R} < \frac{d\sigma_{yp}}{dT} < 0 \end{array} \right\}$	Case E $\left\{ \begin{array}{c} \gamma + \beta \ge 1 \\ \frac{d\sigma_{yp}}{dT} \le \frac{-E\alpha}{1+R} \end{array} \right\}$	Case F $\left\{ \begin{array}{c} \gamma + \beta \ge 1 \\ \frac{-E\alpha}{1+R} < \frac{d\sigma_{yp}}{dT} < 0 \end{array} \right\}$
ELASTIC	$\frac{E\alpha\Delta T}{\sigma_{T_I}} < \frac{1+R}{(1-\beta) - \frac{\lambda(1+R)}{E\alpha}}$	$\frac{E\alpha\Delta T}{\sigma_{T_I}} < \frac{1+R}{(1-\beta) - \frac{\lambda(1+R)}{E\alpha}}$	$\frac{E\alpha\Delta T}{\sigma_{T_I}} < \frac{1+R}{(1-\beta) - \frac{\lambda(1+R)}{E\alpha}}$ R made sufficiently small to satisfy Equation(27).	$\frac{E\alpha\Delta T}{\sigma_{T_I}} < \frac{1+R}{(1-\beta) - \frac{\lambda(1+R)}{E\alpha}}$ $\frac{E\alpha\Delta T}{\sigma_{T_I}} < \frac{1+R}{\gamma}$ R made sufficiently small to satisfy Equation (27).
SHAKEDOWN	$\frac{E\alpha\Delta T}{\sigma_{T_I}} > \frac{1+R}{(1-\beta) - \frac{\lambda(1+R)}{E\alpha}}$ $\frac{E\alpha\Delta T}{\sigma_{T_I}} < \frac{1+R}{\frac{(\gamma-\beta)}{2} - \frac{\lambda(1+R)}{E\alpha}}$ R made sufficiently small to satisfy Equation (21).	$\frac{E\alpha\Delta T}{\sigma_{T_I}} > \frac{1+R}{(1-\beta) - \frac{\lambda(1+R)}{E\alpha}}$ $\frac{E\alpha\Delta T}{\sigma_{T_I}} < \frac{2(1+R)}{(1+\gamma-\beta) - \frac{\lambda(1+R)}{E\alpha}}$ R made sufficiently small to satisfy Equation (21).	$\frac{E\alpha\Delta T}{\sigma_{T_I}} > \frac{1+R}{(1-\beta) - \frac{\lambda(1+R)}{E\alpha}}$ $\frac{E\alpha\Delta T}{\sigma_{T_I}} < \frac{1+R}{\frac{(\gamma-\beta)}{2} - \frac{\lambda(1+R)}{E\alpha}}$ R made sufficiently small to satisfy Equation(21).	$\frac{E\alpha\Delta T}{\sigma_{T_I}} > \frac{1+R}{(1-\beta) - \frac{\lambda(1+R)}{E\alpha}}$ $\frac{E\alpha\Delta T}{\sigma_{T_I}} < \frac{2(1+R)}{(1+\gamma-\beta) - \frac{\lambda(1+R)}{E\alpha}}$ R made sufficiently small to satisfy Equation (21). $\frac{E\alpha\Delta T}{\sigma_{T_I}} > \frac{1+R}{\gamma}$ $\frac{E\alpha\Delta T}{\sigma_{T_I}} < \frac{2(1+R)}{(1+\gamma-\beta) - \frac{\lambda(1+R)}{E\alpha}}$ R made sufficiently small to satisfy Equation (27).
ALTERNATE PLASTICITY	$\frac{E\alpha\Delta T}{\sigma_{T_I}} > \frac{1+R}{\frac{(\gamma-\beta)}{2} - \frac{\lambda(1+R)}{E\alpha}}$ R made sufficiently small to satisfy Equation (23).	$\frac{E\alpha\Delta T}{\sigma_{T_I}} > \frac{2(1+R)}{(1+\gamma-\beta) - \frac{\lambda(1+R)}{E\alpha}}$ R made sufficiently small to satisfy Equation (26).	$\frac{E\alpha\Delta T}{\sigma_{T_I}} > \frac{1+R}{\frac{(\gamma-\beta)}{2} - \frac{\lambda(1+R)}{E\alpha}}$ R made sufficiently small to satisfy Equation(23).	$\frac{E\alpha\Delta T}{\sigma_{T_I}} > \frac{2(1+R)}{(1+\gamma-\beta) - \frac{\lambda(1+R)}{E\alpha}}$ R made sufficiently small to satisfy Equation (26).

CHAPTER V

ANALYSIS OF THE STRUCTURAL BEHAVIOR OF PLATES

The primary purpose of the stress analysis of Chapter III and the analysis of the two-bar model of Chapter IV is to provide information which can be used to predict the resulting structural behavior of a plate subjected to cyclic thermal loading. Therefore, to arrive at some meaningful results which can be used for this purpose and also to be as general as possible, the effect of the following variables on the stress distributions will be investigated. The variables are:

1. Temperature range of the surface of the plate.
2. Rate of heating the surface of the plate.
3. Frequency and shape of the temperature cycle.
4. Yield stress-temperature relationship.

A complete analysis of this problem is essentially two fold. First, the possible structural behavior which can result when a free plate is subjected to cyclic thermal loading on its surfaces must be thoroughly investigated and determined; especially important is how the occurrence of the modes of behavior are affected by the values of the above variables. Then, a method of analysis for predicting the various modes of behavior must be developed.

The modes of behavior have been clearly defined in general for any structure subjected to cyclic thermal loading in Chapter I. One should not construe from this that all structures will undergo each one of these types of behavior, for example, one can refer to the discussion of Payne's paper [6] in Chapter I where he concluded that the T-section would not undergo incremental collapse. The modes of behavior will now be developed and defined specifically for the plate. For clarity to the reader, the investigation is divided into three main divisions. The analysis is greatly simplified at first by assuming that the yield stress is constant, for this allows the development to be strictly dependent upon the variation in the stress distribution through a cross section of the plate. Once the method of analysis for the elastic, shakedown and alternate plasticity modes of behavior are developed for a constant yield strength, a similar analysis will be developed while assuming that the yield stress linearly decreases with temperature. The investigatin of the incremental collapse mode of behavior of the plate is completely different from that of the other three modes and therefore is taken up separately. Nevertheless, this mode of behavior is also presented for the same constant and linearly varying yield stress conditions.

Structural Behavior with Constant Yield Strength

The first step in analyzing the structural behavior of the plate is to observe the stress distributions occurring when the plate is subjected to cyclic thermal loading on its surfaces. A method of stress analysis for this situation has been presented in Chapter III and an example is shown in Appendix B. One can observe in Figure 43 that the maximum tensile and compressive stresses occur at the surface of the plate. One concludes from this that the surface points would be the first to begin yielding as the plate is heated and cooled. Further illustrations of this can also be found in References [10], [11], [12], and [13]. The behavior of the plate will therefore depend upon the stress conditions at its surface. If the surface remains elastic then all other points in the plate will remain elastic. The surface area will be the first to undergo shakedown behavior and alternate plasticity behavior. For instance, if the surface area is in a state of alternate plasticity in each temperature cycle then other points close to the surface could possibly behave in a shakedown manner and still other points further away from the surface would remain elastic. The plate, however, is said to behave in an alternate plasticity mode of behavior.

It is easy to visualize that the behavior of the plate could be approximated by a model consisting of three layers which are fastened together around their circumference as shown in Figure 32, if the temperature of each outer layer is assumed to be at the temperature of the surface of the plate and if the temperature of the center layer is assumed to be at the average temperature of the plate at any time. The two-bar model which was analyzed in Chapter IV is very similar to this three-layer model of the plate; that is, if one considers bar B as the outer two layers and bar A as the center layer, then the behavior of this three-layer model can be obtained from the results of the two-bar analysis. The analysis of the three-layer model would differ from that of the two-bar model in only one way. The stress conditions in the three-layer model would be in a state of "hydrostatic" plane stress; that is, throughout each layer

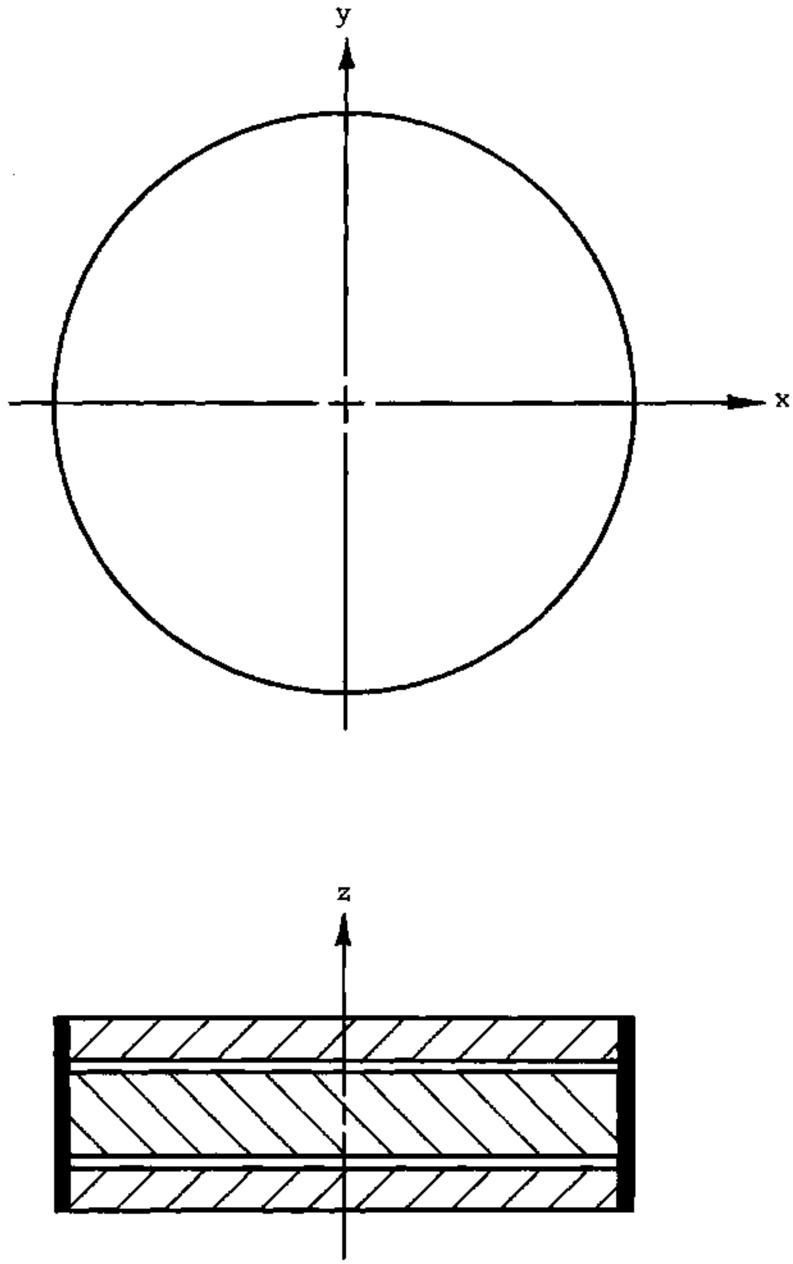


Figure 32. Three-Layer Model of Plate

$$\sigma_x = \sigma_y = \sigma$$

Therefore, the elastic strain in each layer would be written

$$\varepsilon'_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) = \frac{1-\nu}{E}\,\sigma$$

$$\varepsilon'_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) = \frac{1-\nu}{E}\,\sigma$$

Recall that the two-bar analysis was derived under uniaxial stress conditions and the elastic strain was written

$$\varepsilon'_x = \frac{1}{E}\,\sigma_x = \frac{1}{E}\,\sigma$$

Therefore, in order to use the results of the two-bar analysis, one must replace E by $E/(1-\nu)$. With this substitution for E in Figures 13 and 14, the prediction of elastic, shakedown and alternate plasticity behavior of the three-layer model is illustrated in Figures 33 and 34 for the two cases $\gamma+\beta\leq 1$ and $\gamma+\beta\geq 1$, respectively.

Some changes are necessary before the results illustrated in Figures 33 and 34 can be used to predict the behavior of the plate; that is, the terms used in the two-bar analysis will have to be redefined to correspond to the conditions of the plate. Let T_s denote the temperature of the surface of the plate and T_{ave} the average temperature of the plate at any time. From these two sets of values, the following quantities are easily redefined and obtained:

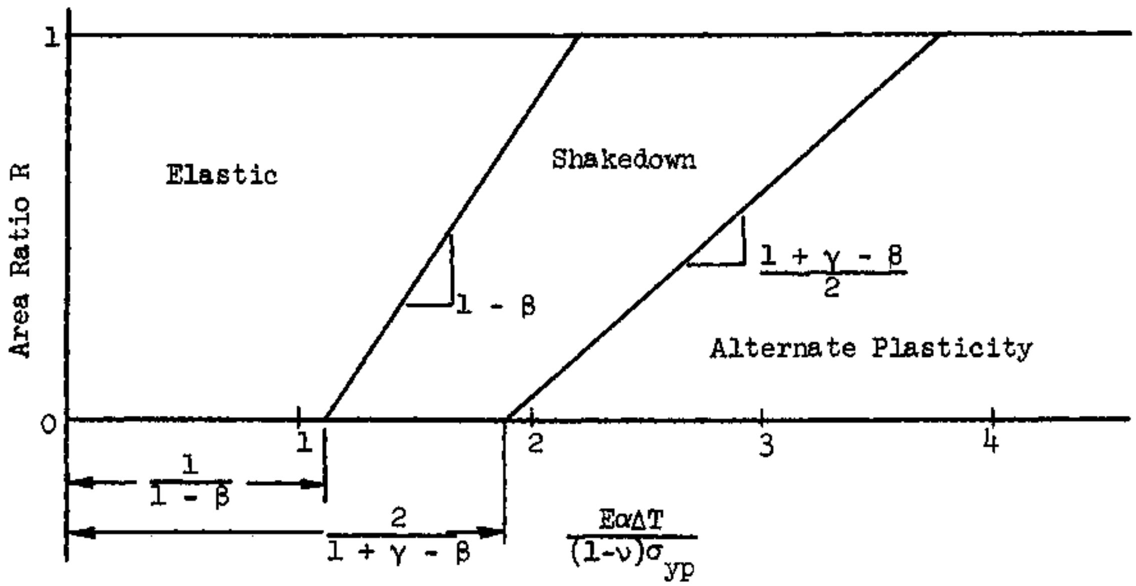


Figure 33. Zones of Behavior of the Three-Layer Model for $\gamma+\beta\leq 1$

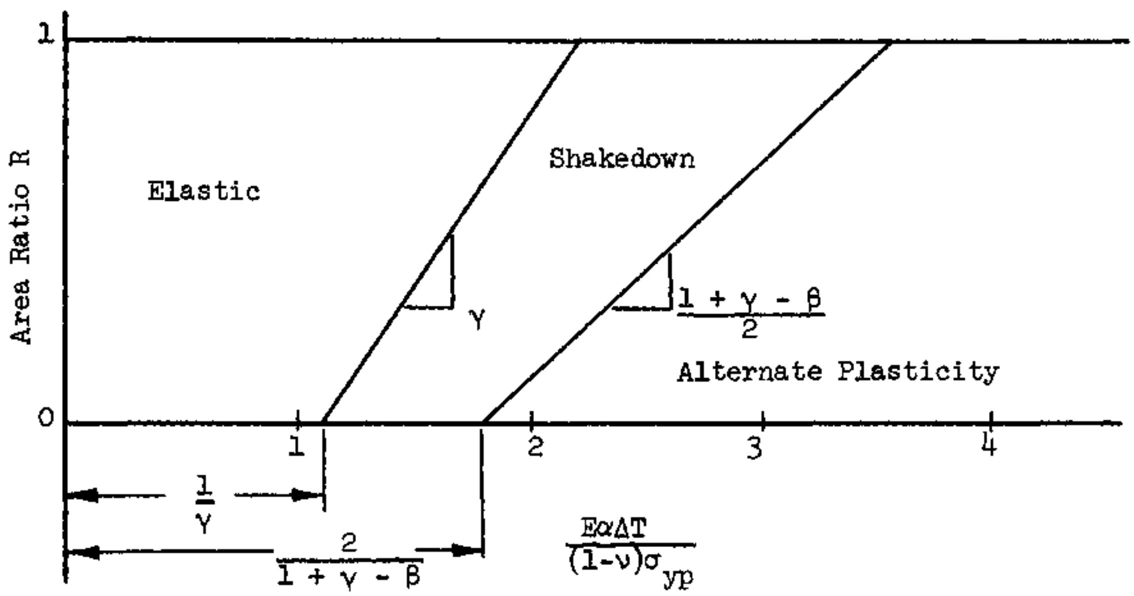


Figure 34. Zones of Behavior of the Three-Layer Model for $\gamma+\beta\geq 1$

T_I = the temperature of the surface of the plate at the time in the temperature cycle when $(T_s - T_{ave})$ is a minimum algebraically.

T_{II} = the temperature of the surface of the plate at the time in the temperature cycle when $(T_s - T_{ave})$ is a maximum algebraically.

$\Delta T = T_{II} - T_I$

$\beta = \dfrac{T_{ave} - T_I}{\Delta T}$, where T_{ave} is measured at the same time in the temperature cycle when T_{II} is measured.

$\gamma = \dfrac{T_{ave} - T_I}{\Delta T}$, where T_{ave} is measured at the same time in the temperature cycle when T_I is measured.

ΔT is equivalent to a measurement of the temperature range of the surface of the plate. A mathematical representation of the periodic time or frequency of the temperature cycle as well as the rate of heating the surface or shape of the temperature cycle is expressed in the form of the two variables γ and β.

R, the ratio of the cross sectional area of bar B to bar A in the two-bar analysis will be redefined for the plate as the ratio of the thickness over which the surface temperature T_s is assumed to penetrate to the remaining thickness of the plate. It is not at all clear at present how to obtain the value or values of R; however, further discussion on this matter will be found later in this chapter.

The type of behavior which results when a plate is subjected to cyclic thermal loading can be determined from Figures 33 and 34 if the material properties E, σ_{yp}, ν and α, and the four variables ΔT, γ, β, and R are known. ΔT, γ and β have been previously defined and are easily obtained by one of the known methods of heat transfer analysis;

however, nothing has been said about determing the value of the area ratio R. This will be discussed next along with the verification that the two-bar model analysis actually yields better predictions for the plate's behavior than obtainable from a numerical solution of the elastoplastic thermal stress Equation (9) in Chapter III.

Verification of Two-Bar Model Results and Determination of R

The two-bar model results will now be verified by spot checking the predicted behavior illustrated in Figures 33 and 34 with the calculated results of some stress analysis solutions similar to the example problem illustrated in Appendix B. After the heat transfer calculations are completed for the particular heating cycle in question and the values of γ and β are determined according to their definitions, one can decide which figure (Figure 33 or 34) should be used. In the stress analysis solution, the integrals in the stress Equation (9) are solved numerically by the trapezoidal rule as described in Appendix B. This is equivalent to assuming that the surface temperature T_s penetrates $\Delta Z/2$ from the surfaces of the plate, where ΔZ is the length of the subintervals. If the numerical procedure uses 11 evenly-spaced points, then

$$Z = \frac{1}{10}$$

or if it uses 21 evenly-spaced points, then

$$Z = \frac{1}{20}$$

Therefore, by definition

$$R = \frac{\frac{1}{10}}{1 - \frac{1}{10}} = \frac{1}{9}$$

and

$$R = \frac{\frac{1}{20}}{1 - \frac{1}{20}} = \frac{1}{19}$$

respectively. By increasing the value of the range of the surface temperature, i.e., increasing ΔT, and solving the stress analysis Equation (9) numerically with the two subinterval spacings mentioned, one can observe at what value of $\frac{E\alpha\Delta T}{(1-\nu)\sigma_{yp}}$ shakedown behavior begins and at what value alternate plasticity behavior begins. Table 4 gives some results from this analysis and also gives the predicted values of $\frac{E\alpha\Delta T}{(1-\nu)\sigma_{yp}}$ obtained from either Figure 33 or 34. As can be seen, the two-bar analysis predictions are extremely accurate. But which answer is correct, the one with R equal to (1/9) or the one with R equal to (1/19)?

The greater the number of points used in a numerical integration of a definite integral, the more accurate the solution becomes; therefore one would conclude that the solution with 21 points is more accurate than the 11 point solution. If more points can be taken, then greater accuracy can be obtained. As the number of steps approaches infinity, the numerical integration becomes identical to an integral solution. This is however impossible to carry out on a digital computer

Table 4. Comparison of the Two-Bar Analysis Predictions with the Stress Analysis Results

(σ_{yp}=Constant)		$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{yp}}$	
		Stress Analysis Results	Predictions from Two-Bar Analysis
SHAKEDOWN BEGINS	$R = \frac{1}{9}$	2.136	2.13675
	$R = \frac{1}{19}$	2.124	2.12500
ALTERNATE PLASTICITY BEGINS	$R = \frac{1}{9}$	2.446	2.44500
	$R = \frac{1}{19}$	2.436	2.43665

as the machine time required would be too great. Notice, however, that R approaches zero as the number of steps in the numerical integration approaches infinity. One conclues from this that the two-bar analysis predictions of the plate behavior become more accurate as R approaches zero. Therefore, taking the limit of the two-bar analysis results of Figures 33 and 34 as R approaches zero, the results are presented in Table 5.

If one considers the problem purely from a physical viewpoint, then one also concludes that R should be zero. That is, in general the slope of the temperature distribution curve is not zero at the surfaces of the plate; therefore, T_s only exist on the surface and, by definition, R equals zero.

Table 5. Conditions Defining the Modes of Behavior of the Plate in Cases A and B

	Elastic	Shakedown	Alternate Plasticity
CASE A		$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{yp}} > \frac{1}{1-\beta}$	
$\gamma + \beta \leq 1$	$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{yp}} < \frac{1}{1-\beta}$		$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{yp}} > \frac{2}{1+\gamma-\beta}$
σ_{yp}=Constant		$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{yp}} < \frac{2}{1+\gamma-\beta}$	
CASE B		$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{yp}} > \frac{1}{\gamma}$	
$\gamma + \beta \geq 1$	$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{yp}} < \frac{1}{\gamma}$		$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{yp}} > \frac{2}{1+\gamma-\beta}$
σ_{yp}=Constant		$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{yp}} < \frac{2}{1+\gamma-\beta}$	

Summary

Given the values of the material properties E, ν, α and σ_{yp}, and the heat transfer variables γ, β and ΔT for a particular cyclic thermally loaded plate, one can determine what type structural behavior will result by use of the appropriate equation or equations in Table 5. This is actually a prediction of the behavior of the surface of the plate; the surface being the worst stressed point in any cross section.

Structural Behavior with Variation of Yield Strength with Temperature

The analysis of the structural behavior of the plate in this section is similar to that in the preceding section except that now the yield stress will be assumed to decrease linearly with temperature. Again the two-bar analysis of Chapter IV will be used to predict the behavior of the plate; also, the new definitions and derivation changes made in the previous section will again be applied here. Recall that since the plate is in a state of plane stress, E in the two-bar analysis must be replaced by $E/(1-\nu)$. By making this substitution in Table 3 and recalling the new definitions given in the last section of the other symbols, the two-bar analysis predictions of the plate behavior are presented in Table 6.

The prediction results can be verified by solving a particular problem by the stress analysis method of Chapter III, for instance the example problem of Appendix B with the yield stress varying with temperature, and comparing the results with the corresponding part of Table 6. As discussed in the last section, the integrals in the stress analysis solution are solved numerically by the trapezoidal rule for 11 and also 21 evenly-spaced points. Hence, in order to compare results, the area ratio R of the two-bar analysis predictions must be made (1/9) and (1/19), respectively. The results of this comparison are shown in Table 7, which illustrates again the extreme accuracy which is obtained from the two-bar analysis predictions.

As the number of points in the numerical solution approaches infinity, the numerical integration becomes identical to an integral

Table 6. Conditions Defining the Modes of Behavior of the Plate in Cases C, D, E, and F for a Linear Yield Stress-Temperature Relationship

	CASE C	CASE D	CASE E	CASE F
	$\gamma + \beta \le 1$ $\frac{d\sigma_{yp}}{dT} \le \frac{-E\alpha}{(1+R)(1-\nu)}$	$\gamma + \beta \le 1$ $\frac{-E\alpha}{(1-\nu)(1+R)} < \frac{d\sigma_{yp}}{dT} < 0$	$\gamma + \beta \ge 1$ $\frac{d\sigma_{yp}}{dT} \le \frac{-E\alpha}{(1+R)(1-\nu)}$	$\gamma + \beta \ge 1$ $\frac{-E\alpha}{(1-\nu)(1+R)} < \frac{d\sigma_{yp}}{dT} < 0$
ELASTIC	$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} < \frac{1+R}{(1-\beta) - \frac{\lambda(1+R)}{\alpha E/(1-\nu)}}$	$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} < \frac{1+R}{(1-\beta) - \frac{\lambda(1+R)}{\alpha E/(1-\nu)}}$	$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} < \frac{1+R}{(1-\beta) - \frac{\lambda(1+R)}{\alpha E/(1-\nu)}}$ R made sufficiently small to satisfy Equation (27).	$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} < \frac{1+R}{(1-\beta) - \frac{\lambda(1+R)}{\alpha E/(1-\nu)}}$ $\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} < \frac{1+R}{\gamma}$ R made sufficiently small to satisfy Equation (27).
SHAKEDOWN	$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} > \frac{1+R}{(1-\beta) - \frac{\lambda(1+R)}{E/(1-\nu)}}$ $\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} < \frac{1+R}{\frac{(\gamma-\beta)}{2} - \frac{\lambda(1+R)}{E/(1-\nu)}}$ R made sufficiently small to satisfy Equation (21).	$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} > \frac{1+R}{(1-\beta) - \frac{\lambda(1+R)}{E/(1-\nu)}}$ $\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} < \frac{2(1+R)}{(1+\gamma-\beta) - \frac{\lambda(1+R)}{E/(1-\nu)}}$ R made sufficiently small to satisfy Equation (21).	$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} > \frac{1+R}{(1-\beta) - \frac{\lambda(1+R)}{E/(1-\nu)}}$ $\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} < \frac{1+R}{\frac{(\gamma-\beta)}{2} - \frac{\lambda(1+R)}{E/(1-\nu)}}$ R made sufficiently small to satisfy Equation (21).	$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} > \frac{1+R}{(1-\beta) - \frac{\lambda(1+R)}{E/(1-\nu)}}$ $\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} < \frac{2(1+R)}{(1+\gamma-\beta) - \frac{\lambda(1+R)}{E/(1-\nu)}}$ R made sufficiently small to satisfy Equation (21). *or* $\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} > \frac{1+R}{\gamma}$ $\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} < \frac{2(1+R)}{(1+\gamma-\beta) - \frac{\lambda(1+R)}{E/(1-\nu)}}$ R made sufficiently small to satisfy Equation (27).
ALTERNATE PLASTICITY	$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} > \frac{1+R}{\frac{(\gamma-\beta)}{2} - \frac{\lambda(1+R)}{E/(1-\nu)}}$ R made sufficiently small to satisfy Equation (23).	$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} > \frac{2(1+R)}{(1+\gamma-\beta) - \frac{\lambda(1+R)}{E/(1-\nu)}}$ R made sufficiently small to satisfy Equation (26).	$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} > \frac{1+R}{\frac{(\gamma-\beta)}{2} - \frac{\lambda(1+R)}{E/(1-\nu)}}$ R made sufficiently small to satisfy Equation (23).	$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} > \frac{2(1+R)}{(1+\gamma-\beta) - \frac{\lambda(1+R)}{E/(1-\nu)}}$ R made sufficiently small to satisfy Equation (26).

solution. In connection with this and also from a purely physical viewpoint, as discussed in the previous section, the limit of the results of the two-bar analysis in Table 6 as R approaches zero gives the true predictions of the behavior of the plate. These predictions are shown in Table 8.

Table 7. Comparison of the Two-Bar Analysis Predictions with the Stress Analysis Results

Yield Stress Linearly Decreases with Temperature		$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}}$	
		Stress Analysis Results	Predictions from Two-Bar Analysis
SHAKEDOWN BEGINS	$R = \frac{1}{9}$	2.054	2.05385
	$R = \frac{1}{19}$	2.027	2.02710
ALTERNATE PLASTICITY BEGINS	$R = \frac{1}{9}$	2.613	2.61277
	$R = \frac{1}{19}$	2.581	2.58056

<u>Summary</u>

If one is given the values of the material properties E, ν, α, the yield stress at temperature T_I (σ_{T_I}) and the slope of the yield stress/temperature curve (λ), and if one is given the heat transfer variables γ, β, T_I, and T_{II} obtained from a heat transfer analysis

Table 8. Conditions Defining the Modes of Behavior of the Plate in Cases C, D, E, and F for a Linear Yield Stress-Temperature Relationship

	CASE C $\gamma + \beta \leq 1$ $\frac{d\sigma_{yp}}{dT} \leq \frac{-E\alpha}{1-\nu}$	CASE D $\gamma + \beta \leq 1$ $\frac{-E\alpha}{1-\nu} < \frac{d\sigma_{yp}}{dT} < 0$	CASE E $\gamma + \beta \geq 1$ $\frac{d\sigma_{yp}}{dT} \leq \frac{-E\alpha}{1-\nu}$	CASE F $\gamma + \beta \geq 1$ $\frac{-E\alpha}{1-\nu} < \frac{d\sigma_{yp}}{dT} < 0$
ELASTIC	$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} < \frac{1}{(1-\beta) - \frac{\lambda(1-\nu)}{E\alpha}}$	$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} < \frac{1}{(1-\beta) - \frac{\lambda(1-\nu)}{E\alpha}}$	$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} < \frac{1}{(1-\beta) - \frac{\lambda(1-\nu)}{E\alpha}}$	$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} < \frac{1}{(1-\beta) - \frac{\lambda(1-\nu)}{E\alpha}}$ $\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} < \frac{1}{\gamma}$
SHAKEDOWN	$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} > \frac{1}{(1-\beta) - \frac{\lambda(1-\nu)}{E\alpha}}$ $\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} < \frac{1}{\frac{(\gamma-\beta)}{2} - \frac{\lambda(1-\nu)}{E\alpha}}$	$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} > \frac{1}{(1-\beta) - \frac{\lambda(1-\nu)}{E\alpha}}$ $\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} < \frac{2}{(1+\gamma-\beta) - \frac{\lambda(1-\nu)}{E\alpha}}$	$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} > \frac{1}{(1-\beta) - \frac{\lambda(1-\nu)}{E\alpha}}$ $\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} < \frac{1}{\frac{(\gamma-\beta)}{2} - \frac{\lambda(1-\nu)}{E\alpha}}$	$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} > \frac{1}{(1-\beta) - \frac{\lambda(1-\nu)}{E\alpha}}$ $\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} < \frac{2}{(1+\gamma-\beta) - \frac{\lambda(1-\nu)}{E\alpha}}$ or $\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} > \frac{1}{\gamma}$ $\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} < \frac{2}{(1+\gamma-\beta) - \frac{\lambda(1-\nu)}{E\alpha}}$
ALTERNATE PLASTICITY	$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} > \frac{1}{\frac{(\gamma-\beta)}{2} - \frac{\lambda(1-\nu)}{E\alpha}}$	$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} > \frac{2}{(1+\gamma-\beta) - \frac{\lambda(1-\nu)}{E\alpha}}$	$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} > \frac{1}{\frac{(\gamma-\beta)}{2} - \frac{\lambda(1-\nu)}{E\alpha}}$	$\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} > \frac{2}{(1+\gamma-\beta) - \frac{\lambda(1-\nu)}{E\alpha}}$

of a particular cyclic thermally loaded plate, the mode of behavior of the plate can be determined from the appropriate equation or equations in Table 8.

Incremental Collapse Mode of Behavior

As mentioned earlier, the investigation of the incremental collapse mode of behavior of the plate is entirely different from that of the other three modes of behavior and therefore will be taken up separately. The difference stems from the fact that incremental collapse requires not only the surface particles but every particle in the plate to undergo yielding sometime during each temperature cycle. The phenomena which causes incremental collapse are complex in nature; however, an understanding can be obtained by considering the behavior pattern of the stress distribution during an entire temperature cycle. The stress distribution can be obtained by making use of the stress analysis procedure of Chapter III.

If the range of the surface temperature, ΔT, is increased even higher than that required to produce alternate plasticity in a particular cyclic thermally loaded plate, a new yield zone at the center of the plate will develop which yields for certain intervals of time during the temperature cycle, but always in the opposite sense to the yield zones occurring near the surfaces. As one continues to increase ΔT, the plate continues to behave in the alternate plastic mode of behavior, even though the center is yielding, until every particle in the elastic region which exists between this center yield region and the surface yield region yields at some time during the temperature cycle. This is possible since yielding is taking place in different points at different times during the cycle. When this occurs, the plate may either begin incremental collapsing or it may continue in an alternate plasticity mode of behavior.

Due to the complexity of having three yield zones with the center one yielding in the opposite sense to the other two and of having moving yield zones, resulting in the yielding of different particles at different times during the temperature cycle, it was concluded that the prediction of the incremental collapse mode of behavior cannot be deduced from an analytical approach such as the two-bar analysis, but rather requires a method of analysis which recognizes the complex inelastic behavior response of the plate to all possible variations of stress and temperature, such as the stress analysis procedure of Chapter III. However, incremental collapse cannot be predicted explicitly by this method. First the stress distributions must be obtained through a numerical procedure and then they must be analyzed to see if incremental collapse will possibly result. This analysis will be discussed next for a constant yield stress and then for a yield stress which decreases with temperature.

Incremental Collapse Analysis with Constant Yield Stress

1. The stress and plastic strain distributions are obtained for the particular cyclic thermal stress conditions in question by means of the stress analysis procedure described in Chapter III and by use of a computer program similar to the one given in Appendix C. Since this is a numerical procedure, the stresses and plastic strains are obtained for n number of points through the cross section of the plate and for m

number of time or loading increments during each cycle. The stress distributions must be calculated for at least four cycles and prefereable for six or more cycles.

2. A necessary condition for incremental collapse is that yielding occurs at least at both surfaces and at the center of the cross section.

3. If the necessary condition in No. 2 is satisfied and if there is at least one point between the surface yield zone and the center yield zone which does not yield at any time in at least the first four cycles, then one can determine if incremental collapse will occur by the following procedure:

(a) Determine from the stress distribution for the first complete cycle the point in the elastic region which has the smallest maximum stress (tension or compression), that is, the point which would be the last to yield.

(b) Further inspection will determine if this maximum stress at this point remains constant or increases in subsequent cycles. If it remains constant then incremental collapse will not occur; however, if it increases then it will increase by a smaller amount each cycle and extrapolation for additional cycles will indicate whether or not it will cease to increase before it reaches the yield stress. If it reaches the yield stress, then the amount of plastic strain which occurs at this point will reoccur in subsequent cycles, thus producing incremental collapse.

4. If all the points in the cross section yields at any time in the first four cycles, then it is necessary to inspect any one of the points for two or three additional cycles to see if the variation in the plastic strain is perfectly cyclic in nature (repeats identically each cycle) or if it becomes more positive or more negative after each cycle. If it repeats itself each cycle, then alternate plasticity results; however, if it becomes more positive or negative after each cycle, then incremental collapse is occurring.

<u>Incremental Collapse Analysis with Variation of Yield Stress with Temperature</u>

1. The stress and plastic strain distributions are obtained for the particular cyclic thermal stress conditions in question in the same manner as previously described with a constant yield stress except now the yield stress will vary with temperature.

2. Again, the necessary condition for incremental collapse is that yielding occurs at least at both surfaces and at the center of the cross section.

3. If the necessary condition of No. 2 is satisfied and if there is at least one point between the surface yield zone and the center yield zone which does not yield at any time in the first four cycles, then it is necessary to observe each of the points which have not yielded for several additional cycles. If the variation in their stresses is perfectly cyclic in nature (repeats identically each cycle), then incremental collapse will not occur. However, if the variation in

their stresses become more positive or more negative in subsequent cycles, then it is necessary to continue cycling until the stresses begin to repeat identically each cycle or until each point begins to yield (this will usually require less than ten cycles). If each point begins to yield, then incremental collapse will result.

4. If all the points in the cross section yields during the first four cycles then the analysis is the same as part 4 of the constant yield stress case.

Discussion

Incremental collapse behavior of the plate is too complex to be given in equation, table or graph form; therefore, each particular cyclic thermally stressed plate will have to be analyzed individually in order to determine if incremental collapse behavior will result. However, a better understanding can be gained as to the type results which can be expected by considering a particular problem. Consider, for instance, the example problem in Appendix B where the surface temperature is increased and decreased linearly and the heat transfer conditions are such that $\gamma = 0.2817$ and $\beta = 0.5567$. Incremental collapse begins for this case at $\frac{E\alpha\Delta T}{(1-\nu)\sigma_{yp}} = 26.00$ for a constant yield stress and at $\frac{E\alpha\Delta T}{(1-\nu)\sigma_{T_I}} = 10.09$ for a linearly decreasing yield stress $\left(\frac{\lambda(1-\nu)}{E\alpha} = -0.05\right)$. The formation and disappearance of their yield zones for one complete cycle are shown in Figures 35 and 36, respectively. Notice that the region between Z = 0.405 and 0.547 in Figure 35 and between Z = 0.300 and 0.616 in Figure 36 yields only in compression while the rest of the cross section yields in tension as well as in compression.

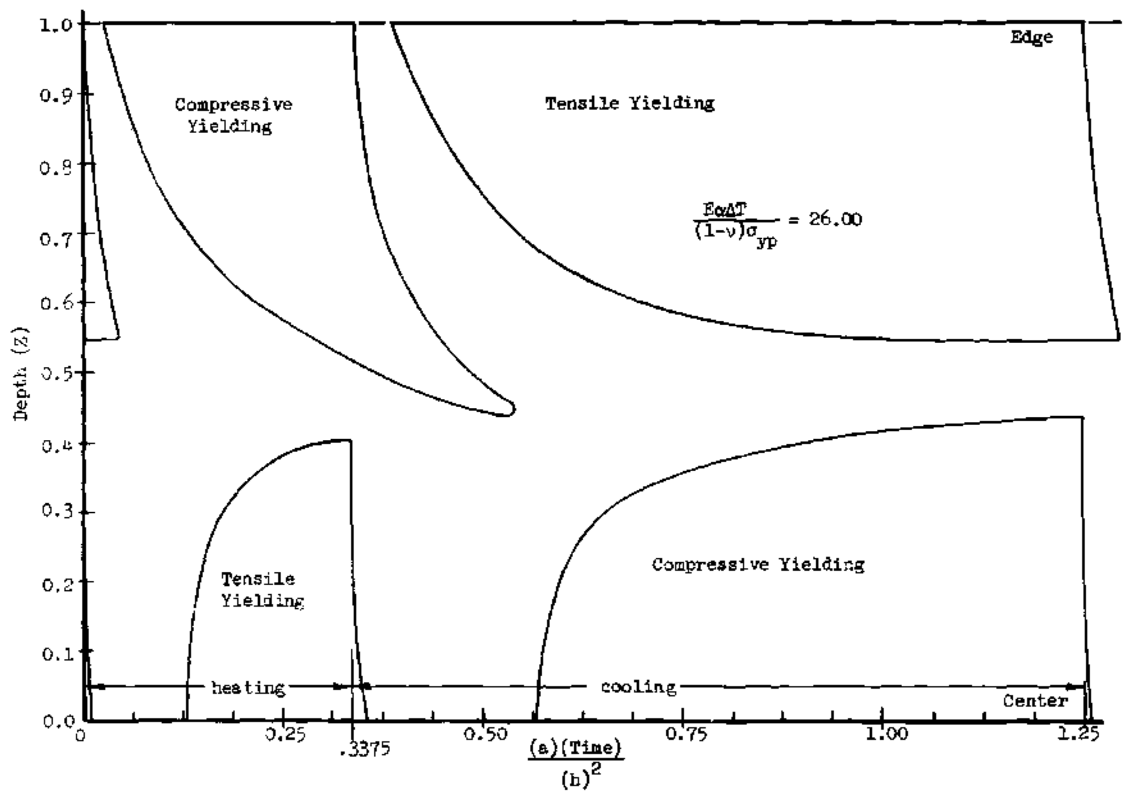


Figure 35. Formation and Disappearance of Yield Zones for One Cycle with a Constant Yield Stress

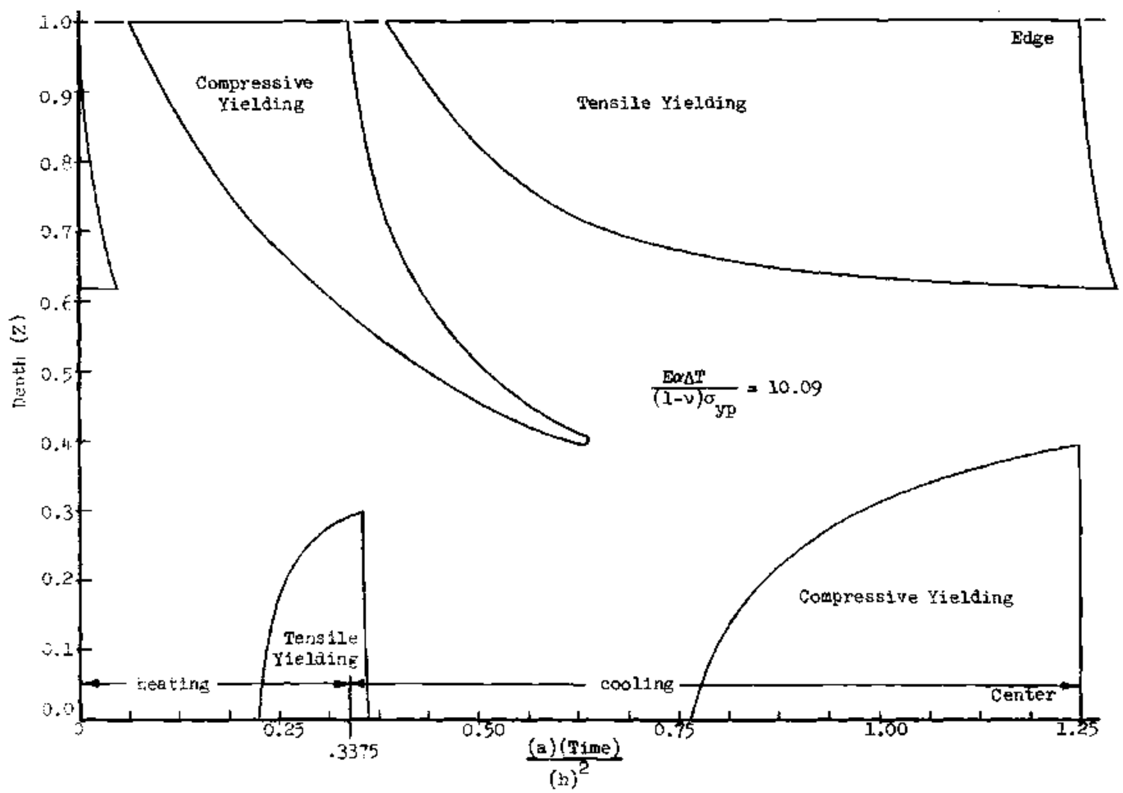


Figure 36. Formation and Disappearance of Yield Zones for One Cycle with the Yield Stress Decreasing with Temperature

This slight compressive yielding will occur again in consecutive cycles resulting in incremental collapse (negative) of the plate. A more thorough analysis of this same type problem follows.

Linearly Varying Surface Temperature Example. Consider now a more general case of the same type problem as the above-mentioned example, i.e., consider the effects of the steady periodic temperature in a plate in which both surface temperatures are varied in the manner shown in Figure 37. In order to be able to compare the incremental collapse predictions of this section to the predictions of elastic, shakedown and alternate plasticity behavior, the same terms (γ, β and ΔT) will be used here. The yield stress is assumed constant.

For this particular type heating and cooling, ΔT is the same as the actual range of the surface temperature, as shown in Figure 37. By changing the values of TIME 1 and PERIOD (defined in Figure 37) the values of γ and β will vary. Predictions of incremental collapse are obtained by means of a trial and error procedure, that is, by step increasing the value of $\frac{E\alpha\Delta T}{(1-\nu)\sigma_{yp}}$ while holding γ and β constant, one can determine by means of the previously discussed procedure the value of $\frac{E\alpha\Delta T}{(1-\nu)\sigma_{yp}}$ at which incremental collapse will begin. Then the values of Period and Time 1 are changed in such a way that either the value of γ or β is varied and the procedure is repeated. The resulting predictions of incremental collapse are shown in Figures 38 and 39 for positive and negative incremental collapse and for a variation in γ and β, respectively. For comparison, these predictions are reproduced in Figures 40 and 41 along with predictions of elastic, shakedown and alternate plasticity behavior obtained from the equations in Table 5.

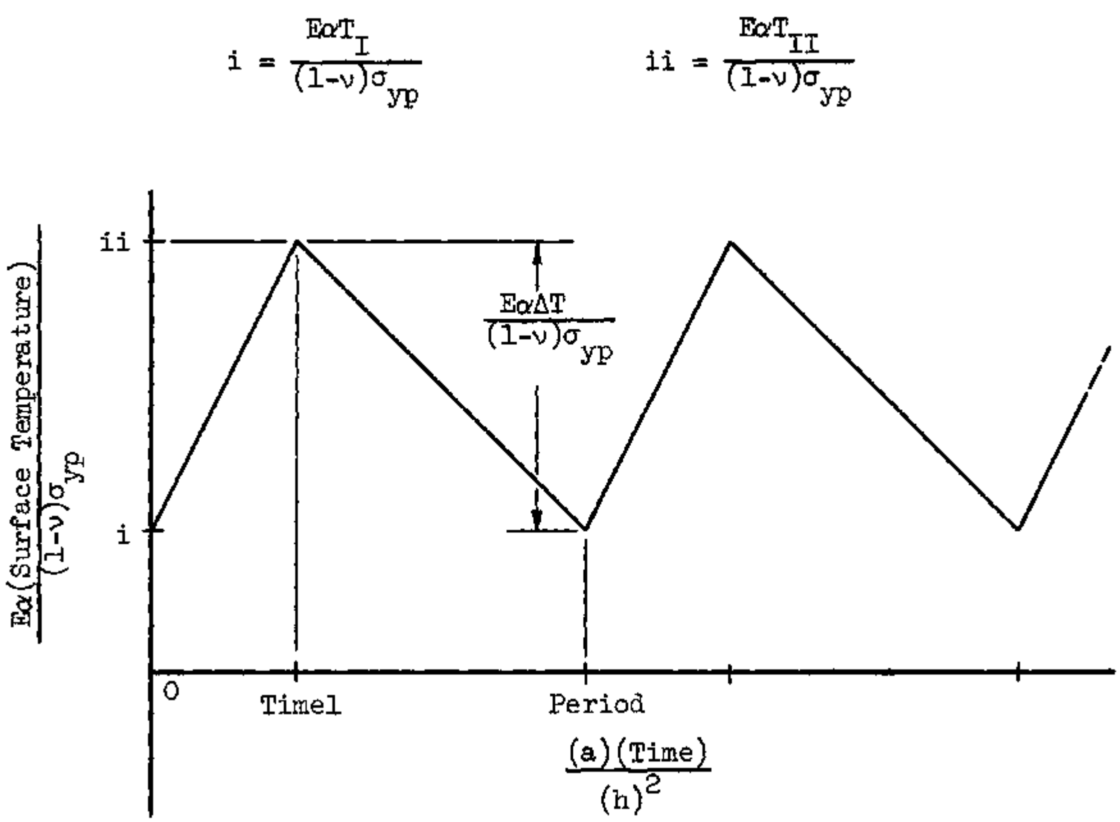


Figure 37. Linearly Varying Surface Temperature

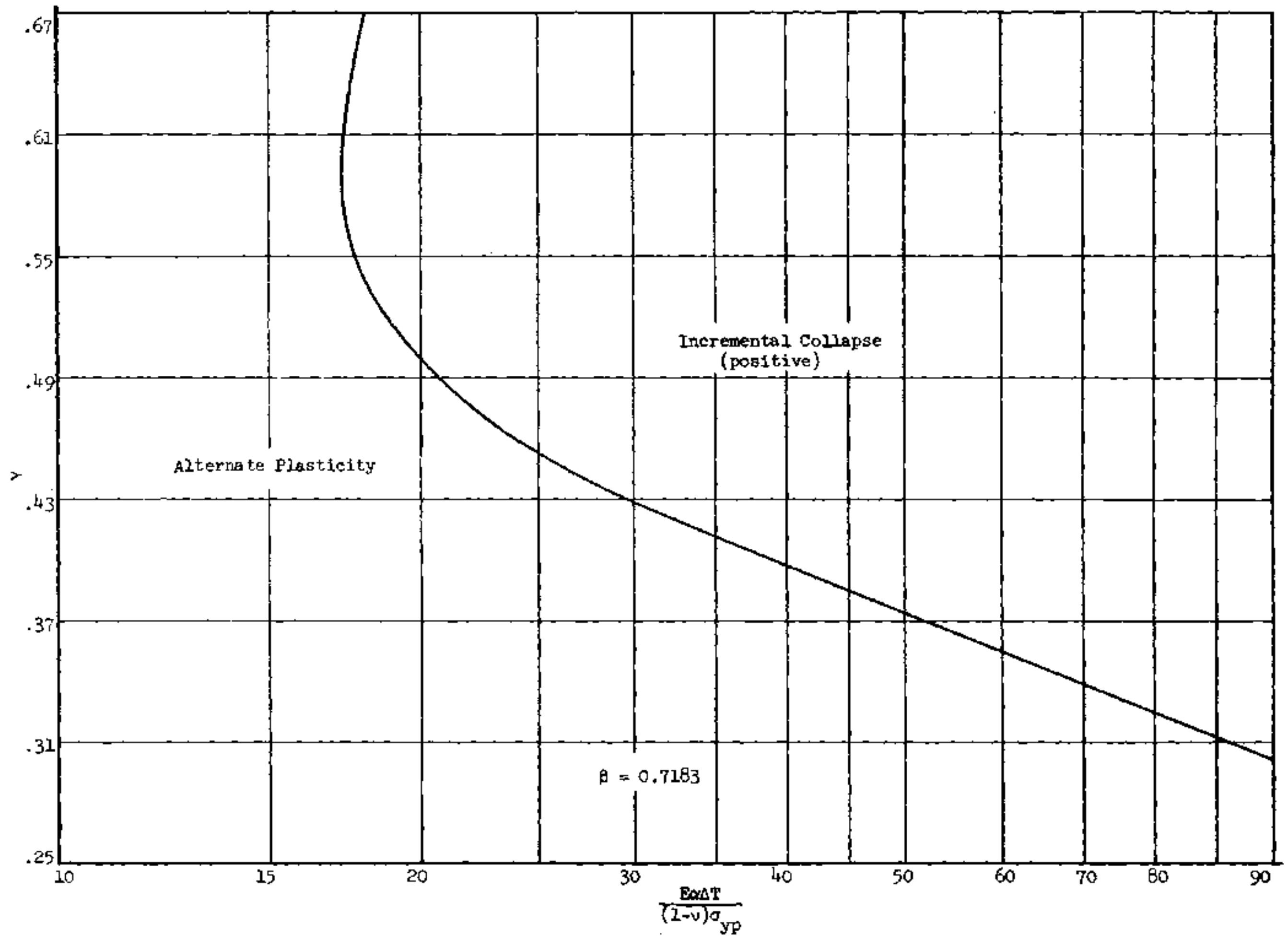


Figure 38. Incremental Collapse Prediction for a Linearly Varying Surface Temperature

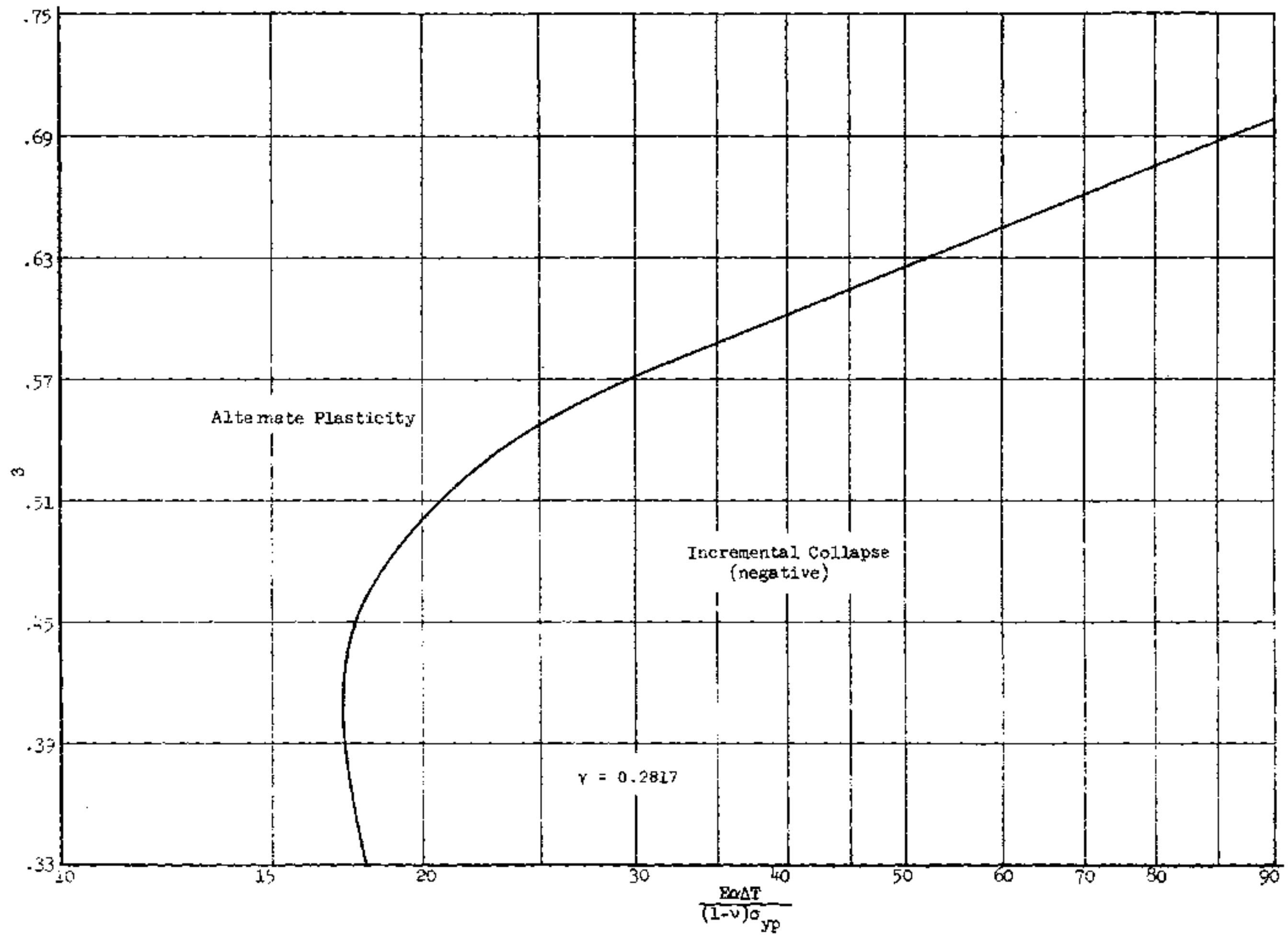


Figure 33. Incremental Collapse Prediction for a Linearly Varying Surface Temperature

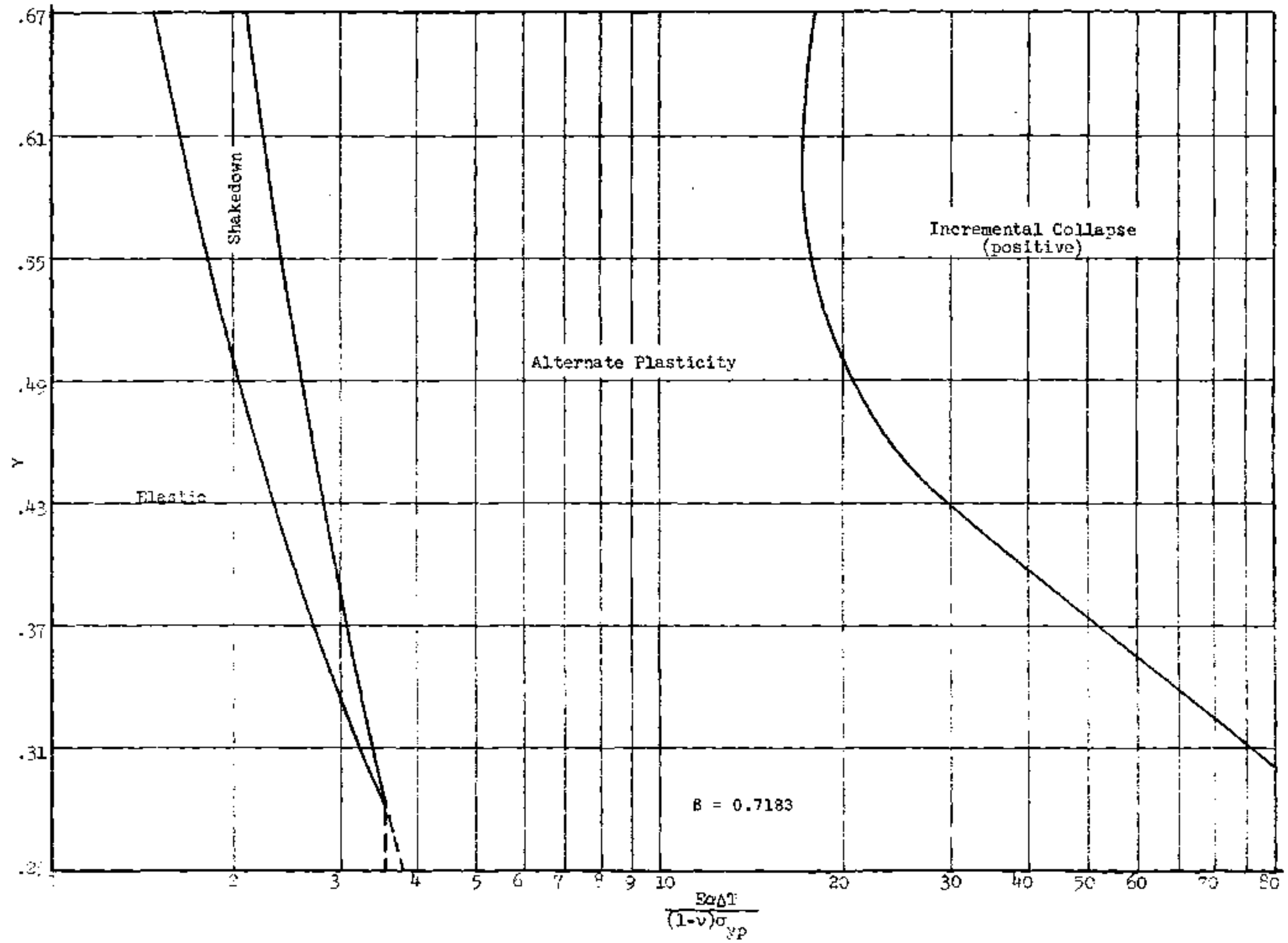


Figure 40. Modes of Behavior for a Linearly Varying Surface Temperature

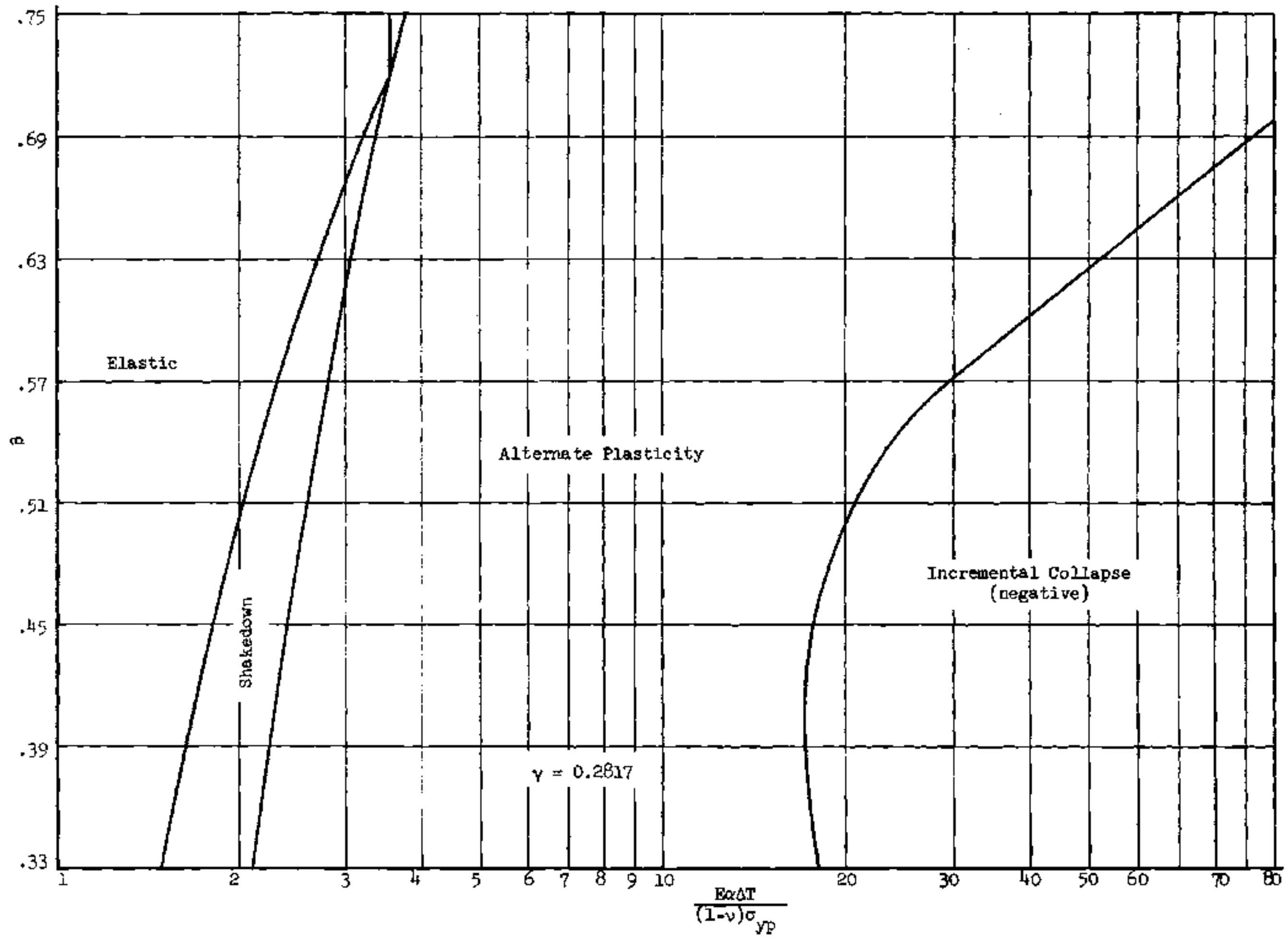


Figure 41. Modes of Behavior for a Linearly Varying Surface Temperature

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER RESEARCH

Conclusions

The theoretical analysis made in this study concludes that a free plate can either deform elastically, shakedown to an elastic state, yield in an alternate plasticity mode of behavior or yield in an incremental collapse mode of behavior when subjected to cyclic thermal loading, even without considering any variation of the properties of the material. A means of analyzing the various modes of behavior which may result has been developed and a method of predicting the behavior of the plate has been presented. For elastic, shakedown and alternate plasticity behavior, predictions were attained explicitly through the development of a two-bar model analysis; however, it was concluded that the prediction of the incremental collapse mode of behavior can not be attained from an analytical approach of this type, but rather requires the stress analysis procedure presented in Chapter III.

In addition to the above conclusions, the following conclusions were also deduced from the results of this study:

1. Elastic, shakedown and alternate plasticity modes of behavior are directly dependent upon the four dimensionless ratios β, γ, $\frac{E\alpha\Delta T}{(1-\nu)\sigma_{yp}}$ and $\frac{\lambda(1-\nu)}{E\alpha}$. β and γ are, in turn, dependent upon the rate of heating, the frequency and shape of the temperature cycle, and the thickness of the plate. If the yield stress is assumed constant

(λ=0) and the heating conditions are such that $\beta+\gamma=1$, then the mode of behavior called shakedown will not occur. This can be seen by comparing the corresponding equations in Table 5. Also, an example is given in Figures 40 and 41 showing this result. The condition of $\beta+\gamma=1$ occurs in cases where there are symmetric heating and cooling half cycles, such as a sine or cosine surface heating function.

2. As noted earlier in this study, plastic regions begin developing on both surfaces of the plate and penetrate inward. Under more severe conditions, such as a larger range of the surface temperature or a higher frequency of the heating cycle, a third plastic region will develop at the median plane of the plate. This center yield zone will always be of opposite sense to the surface yield zones and therefore there will always be an elastic region existing between them. This elastic region, however, will not remain in a static position, but will continuously change size and position during a temperature cycle as shown in Figures 35 and 36. Due to this movement, all particles in the cross section may yield at some time during the cycle and thus it is possible for incremental collapse to occur. Note that the surface region and the center region actually undergoes what might be called a combined alternate plasticity and incremental collapse behavior, for they yield in tension and compression during each cycle, yet their net plastic strain increases with each cycle.

3. Even though the incremental collapse mode of behavior cannot be uniquely defined by the same four dimensionless ratios used in defining the other three modes, they can still be used to qualitatively discuss the results. If the yield stress is assumed constant (λ=0) and

the heating conditions are such that $\gamma+\beta<1$, then the incremental collapse behavior will probably be negative (compressive) and the plate will progressively become smaller. If, however, $\lambda=0$ and $\gamma+\beta>1$, then the incremental collapse behavior will probably be positive (tensile) and the plate will progressively become larger. If on the other hand $\lambda=0$ and $\gamma+\beta=1$, then incremental collapse will probably not occur and only elastic and alternate plasticity behavior will result. This latter condition was also concluded in the paper by Yuksel [11] where he varied the surface temperature harmonically (sine function). If the yield stress is allowed to decrease with temperature ($\lambda<0$), then negative incremental collapse will occur more readily, even occurring when $\gamma+\beta=1$, and positive incremental collapse will be harder to produce.

Recommendations

The following limited recommendations are presented concerning further research on the behavior of structures subjected to cyclic thermal stressing conditions:

1. The approach presented in this study for analyzing the inelastic structural behavior of a free plate subjected to cyclic thermal loading should be extended to also include mechanical stressing conditions. The mechanical stresses might result from edge support conditions or applied loadings which may either be constant or cyclic in nature.

2. The generalized two-bar structural analysis derived in Chapter IV should be used to predict the structural behavior of other structures other than the plate. This approach can be developed

further with respect to the analysis of many built-up complex structures. Also, many simple structures can now be analyzed more fully, for instance, the T-section beam should be analyzed to see if the effect of the heat conduction into the web will produce incremental collapse. The present two-bar analysis should also be extended to include generalized mechanical loading conditions.

3. Under cyclic thermal loading, alternate plasticity behavior may lead to failure by a fatigue process usually referred to as thermal-stress-fatigue. Incremental collapse behavior, however, leads to excessive deformations or to failure by fracture. This study has shown that when a plate is in a state of incremental collapsing, the surface area is actually undergoing reversed plastic flow with a net plastic flow in either compression or tension during each cycle. Life predictions for a plate or structure stressed under these conditions should be formulated.

4. The same analytical method used in this study can be easily used to begin analyzing the effects of material property variations, such as strain hardening or the variation of material properties with temperature for specific heating conditions.

APPENDIX

APPENDIX A

CONVERGENCE OF ITERATION PROCEDURE

A rigorous proof of the convergence of the iteration procedure used to solve either the complicated set of nonlinear equations representing the general three-dimensional elastoplastic problem or the integral Equation (9) developed for the present plane stress problem is beyond the scope of this investigation. This discussion will therefore be limited to qualitative observations and the combined experience of Manson [14] and Mendelson [15] with this type problem.

In the iteration procedure described in Chapter III, the equivalent stress (σ_e) is computed by means of Equation (5), then the equivalent plastic strain increment ($\Delta\varepsilon_e''$) is obtained from the idealized (elastic-perfectly plastic) stress-strain curve of the material at the points where yielding will occur by dropping down along a constant strain line to the curve. This is illustrated in Figure 42 by path OABC. The reason that this procedure will probably converge can be seen qualitatively by observing that since the stress-strain curve is very steep in the elastic portion, a small error in σ_e will produce a smaller error in $\Delta\varepsilon_e''$. Even though Mendelson [15] did not use this same procedure when he illustrated his method of successive elastic solutions, he did present a similar qualitative reason for convergence and also stated that experience has shown that the method will converge, provided the loading increment is made sufficiently small. How small is sufficiently small

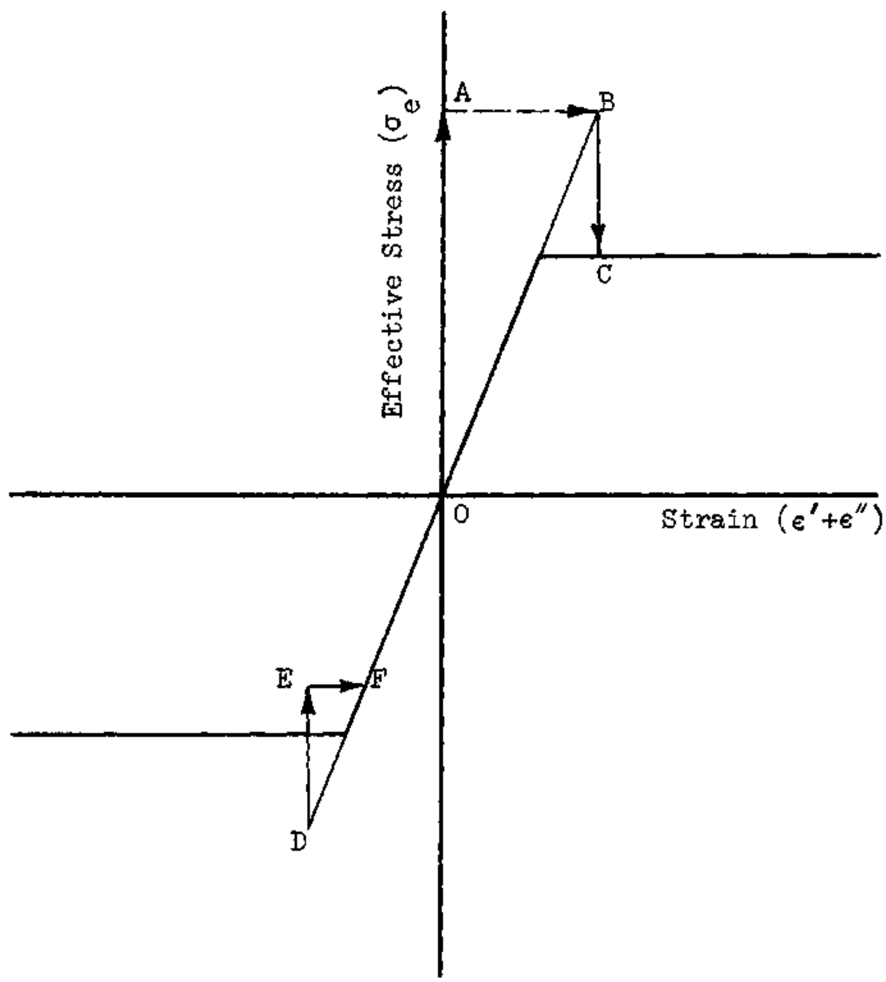


Figure 42. Determination of the Equivalent Plastic Strain Increment ($\Delta\epsilon''_e$) from the Equivalent Stress (σ_e)

for a particular problem can only be determined by trial and error, i.e., if the process diverges, the increment size is reduced.

Manson [14] states that in some cases successive approximations lead to a convergent solution while in other cases the solution diverges; if it converges at all, it will converge to the correct solution; if it diverges, it is easily detected and it is usually possible to make the solution converge by making appropriate changes in the process. However, experience with the present problem has shown that even though the solution will converge, it may converge to the wrong solution. This may be seen by considering two points in a cross section of the plate, one being stressed in tension and the other in compression. Assume that the stress at the first point is calculated to be at point B in Figure 42 before the first step of the iteration procedure reduces it to the yield stress. This elastic solution could cause the second point to be stressed in compression to point D. The iteration procedure will then reduce both stresses to their yield stress by introducing some plastic strain at each point; however, due to the relieving of the tensile stress from point B to point C, the compressive stress could actually fall below the yield stress, say to point E in Figure 42. This shows that the point which was stressed in compression should not have yielded as much as calculated or even possible should not have yielded at all. Of course, if the loading increment is sufficiently small, this will not occur; but the question of how small is sufficiently small for a particular problem is not so easily determined.

This problem was handled in the following manner. If a point yields during an iteration and its stress is then reduced below its yield stress, then the plastic strain which was introduced at that point during that iteration is removed and the iteration is repeated. This would, for example, place the stress-strain state of point E at point F in Figure 42. By adding this procedure to the iteration procedure described in Chapter II and III and using a sufficiently small loading increment to obtain convergence, the procedure will converge to the correct solution.

APPENDIX B

ILLUSTRATION OF STRESS ANALYSIS SOLUTION

To illustrate the application of the solution procedure discussed in Chapter III, an analysis of a plate subjected to cyclic heating on its surfaces in the manner shown in Figure 37 will be described. Since the resulting time-dependent nonlinear temperature distribution through the plate is not easily found in the literature in exactly the form needed here, it was obtained by a finite difference solution of the diffusion equation.

The diffusion equation in one dimension

$$\frac{1}{a}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial z^2}$$

written as a finite difference equation for a finite time increment (Δt) and a dimensionless finite subdivision ($\Delta Z = \Delta z/h$) becomes

$$T_j^{i+1} = (1-P)T_j^i + P\,\frac{T_{j+1}^i + T_{j-1}^i}{2} \qquad (34)$$

where

$$P = \frac{2a\Delta t}{h^2 \Delta Z^2}$$

The superscript i refers to the time increment and the subscript j refers to the subdivision location. In order not to introduce any instabilities in the numerical solution of Equation (34) and possibly violate thermodynamic principles, one must select ΔZ and Δt so that

$$\frac{2a\Delta t}{h^2\Delta Z^2} \leq 1$$

Therefore, an upper limit is placed on the permissible value of the time increment Δt by a choice of ΔZ.

The stress analysis procedure, as described in Chapter III, was necessarily programmed for the Univac 1108 digital computer. Therefore, in order to calculate the resulting stress distributions through the plate, the integrals in Equation (9) are evaluated by a finite difference approximation method as follows. If the plate is divided into n points $(Z_1, Z_2, Z_3, ---, Z_n)$ and the points Z_1 and Z_n are identified with the end points $Z = -1$ and $Z = +1$, respectively, and if a uniform spacing is chosen, so that

$$\Delta Z = \frac{2}{n - 1}$$

one recalls that the Trapezoidal Rule gives

$$\int_{-1}^{1} f(Z)dZ \approx \frac{2}{n-1} \sum_{j=1}^{n} B_j f(Z_j)$$

where

$$(B_1, B_2, B_3, ---, B_{n-1}, B_n) = (\tfrac{1}{2}, 1, 1, ---, 1, \tfrac{1}{2})$$

Therefore, Equation (9) is replaced by the following finite difference equation:

$$\frac{\sigma(Z_j,t)}{\sigma_{yp}} = \frac{1}{n-1} \sum_{j=1}^{n} \left\{ B_j \left[\frac{E\alpha T(Z_j,t)}{(1-\nu)\sigma_{yp}} \right] \right\} - \left[\frac{E\alpha T(Z_j,t)}{(1-\nu)\sigma_{yp}} \right]$$

$$+ \frac{1}{n-1} \sum_{j=1}^{n} \left\{ B_j \left[\frac{E\varepsilon''(Z_j,t)}{(1-\nu)\sigma_{yp}} + \frac{E\Delta\varepsilon''(Z_j,t)}{(1-\nu)\sigma_{yp}} \right] \right\}$$

$$- \left[\frac{E\varepsilon''(Z_j,t)}{(1-\nu)\sigma_{yp}} + \frac{E\Delta\varepsilon''(Z_j,t)}{(1-\nu)\sigma_{yp}} \right]$$

The number of points, n, determines how close the numerical integration approximates the integral solution. For this illustrative example and for most of the other solutions in this report, n was taken as 21, which is believed to be a comparatively large number of points when one considers the overall thickness of most plates.

Since the present method of stress analysis uses an iteration procedure which incrementally steps through time, one must also determine the size of the time increment needed for the solution of each particular problem at hand. Criteria for determining the time increment size are discussed next.

Determination of the Time Increment Size

Two major considerations have been mentioned concerning the length of the time interval used in the incremental solution procedure. In Appendix A it was mentioned that the time or loading increment has to be sufficiently small to assure convergence of the solution. This requirement is easily handled by the method described there and therefore will not be discussed further here. However, convergence is not the only determinator of the increment size. Recall that the basic derivation of the incremental theory of plasticity requires that the increment size be small enough to assure that proportional loading exist during each increment of loading which produces some amount of plastic strain. The accuracy of the solution is greatly dependent upon this assumption; therefore, a suitable means of determining the time or loading increment size which will allow some degree of accuracy from the proposed procedure is needed.

The size of the time increment cannot be made unlimitedly small due to the limitation of the available computer time. In other words, the smaller the size of the time increment, the greater the number of increments in a loading history and therefore the longer the computational time required. Excessive machine time can be reduced by considering uneven time intervals, using smaller intervals in the portions of the cycle where large plastic strains are experienced and using larger intervals in the other portions.

The time intervals used in the present problem were determined by comparing the different values of the stress distributions through the plate obtained from several solutions, each using a smaller time interval, until the stress distributions do not change appreciably with a change in the increment size. If the time intervals could be made small enough so that proportional loading is produced in each increment, the stress distributions would cease to change even if smaller intervals were used. The number of time intervals considered were approximately 100 to 400 increments per cycle. Uniform sized time intervals were used in the present problem since plastic strain was observed in some particle for almost every interval during the cycle for the majority of the cases run.

Results

The results of this illustrative example were obtained by use of a digital computer program (see Appendix C). The resulting dimensionless stress distributions at different times during the cycle are shown in Figures 43, 44, and 45 for a constant yield stress and in Figures 46, 47 and 48 for a linearly decreasing yield stress with temperature, illustrating elastic, shakedown and alternate plasticity behavior of the plate.

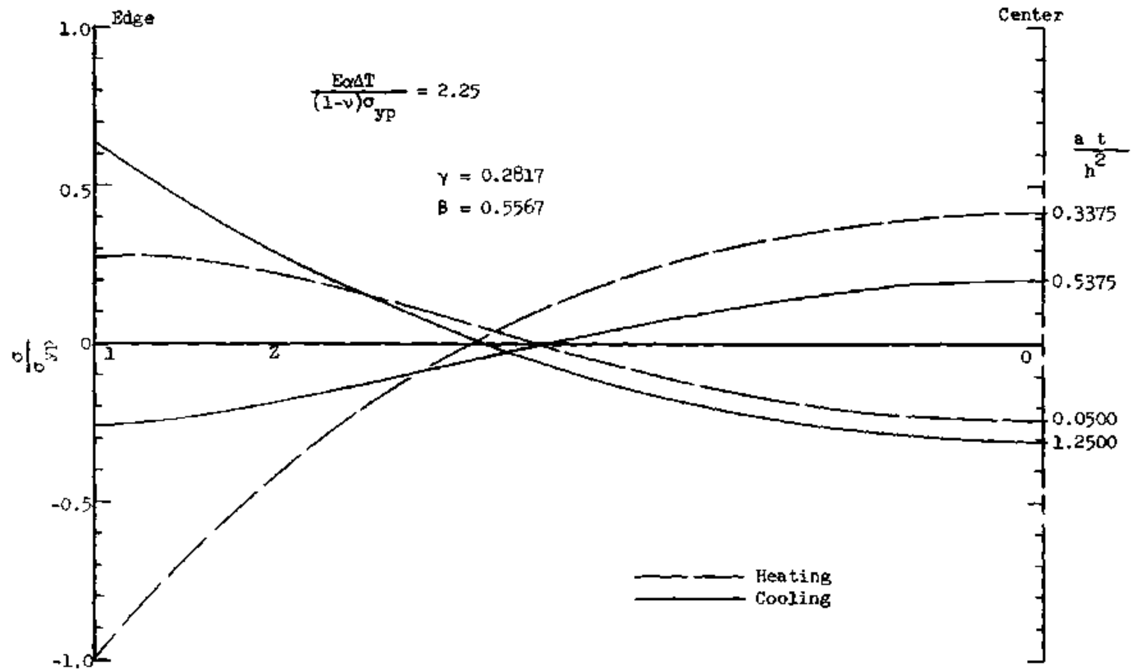


Figure 43. Stress Distributions for One Cycle Illustrating Elastic Behavior of a Plate with a Constant Yield Stress

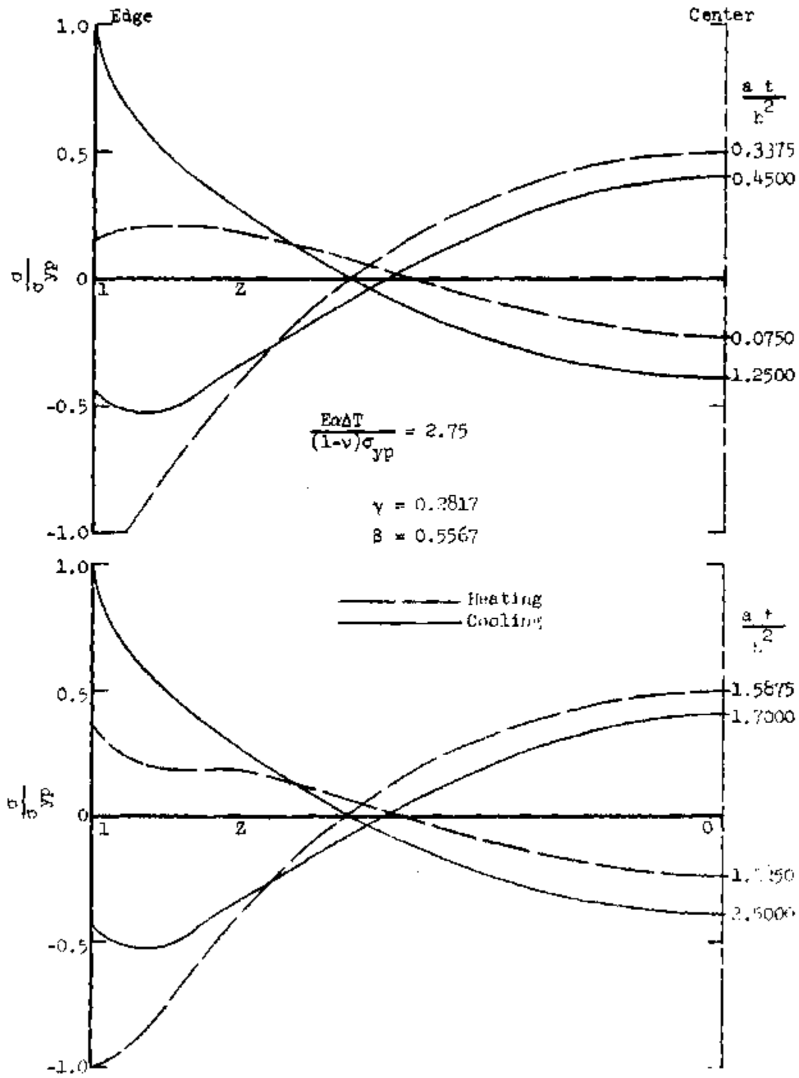


Figure 44. Stress Distributions for the First and Second Cycles Illustrating Shakedown Behavior of a Plate with a Constant Yield Stress

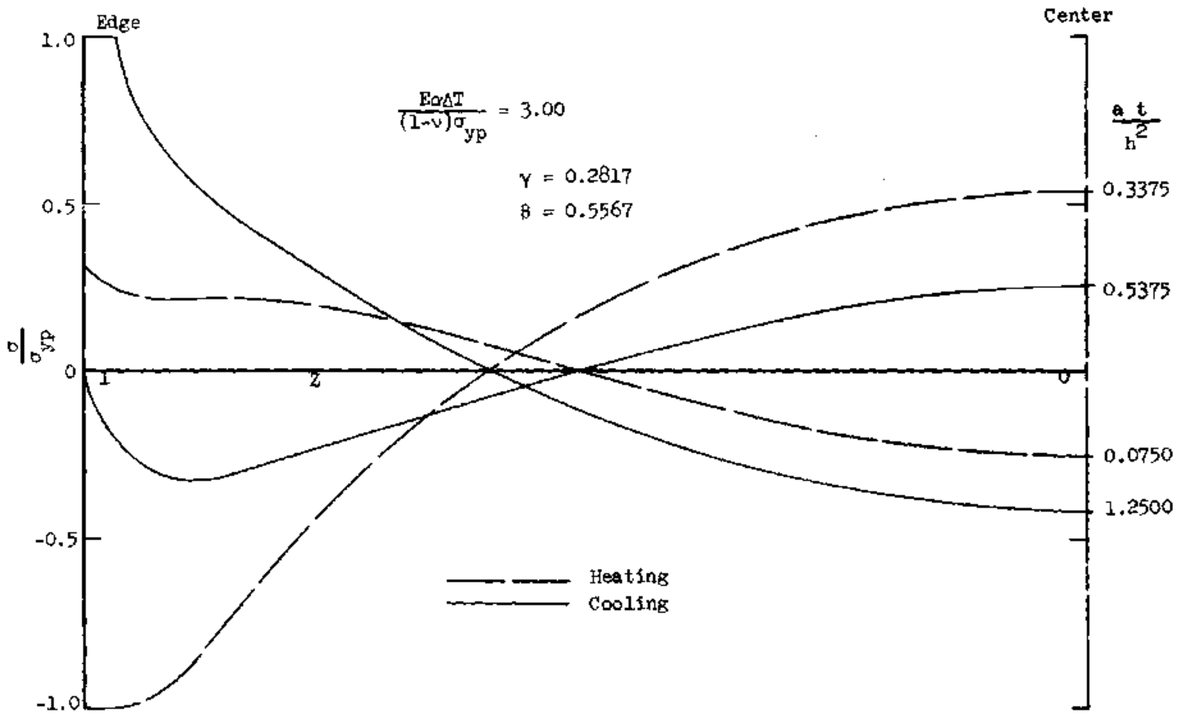


Figure 45. Stress Distributions for One Cycle Illustrating Alternate Plasticity Behavior of a Plate with a Constant Yield Stress

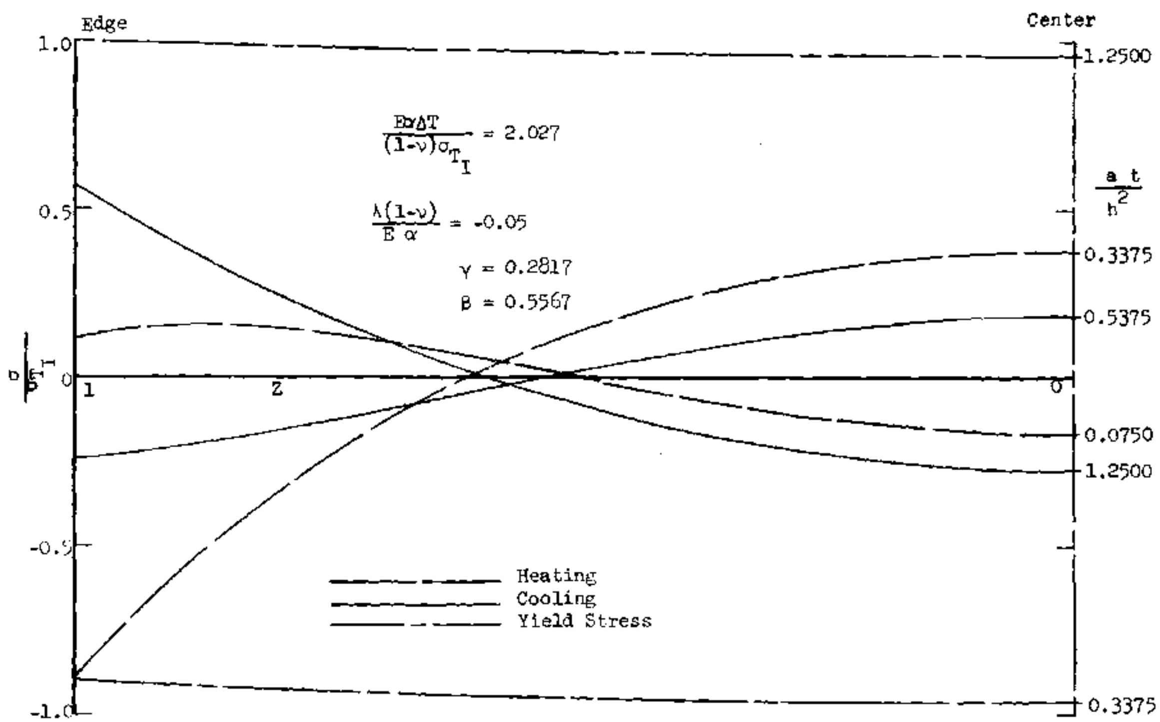


Figure 46. Stress Distributions for One Cycle Illustrating Elastic Behavior of a Plate with the Yield Stress Linearly Decreasing with Temperature

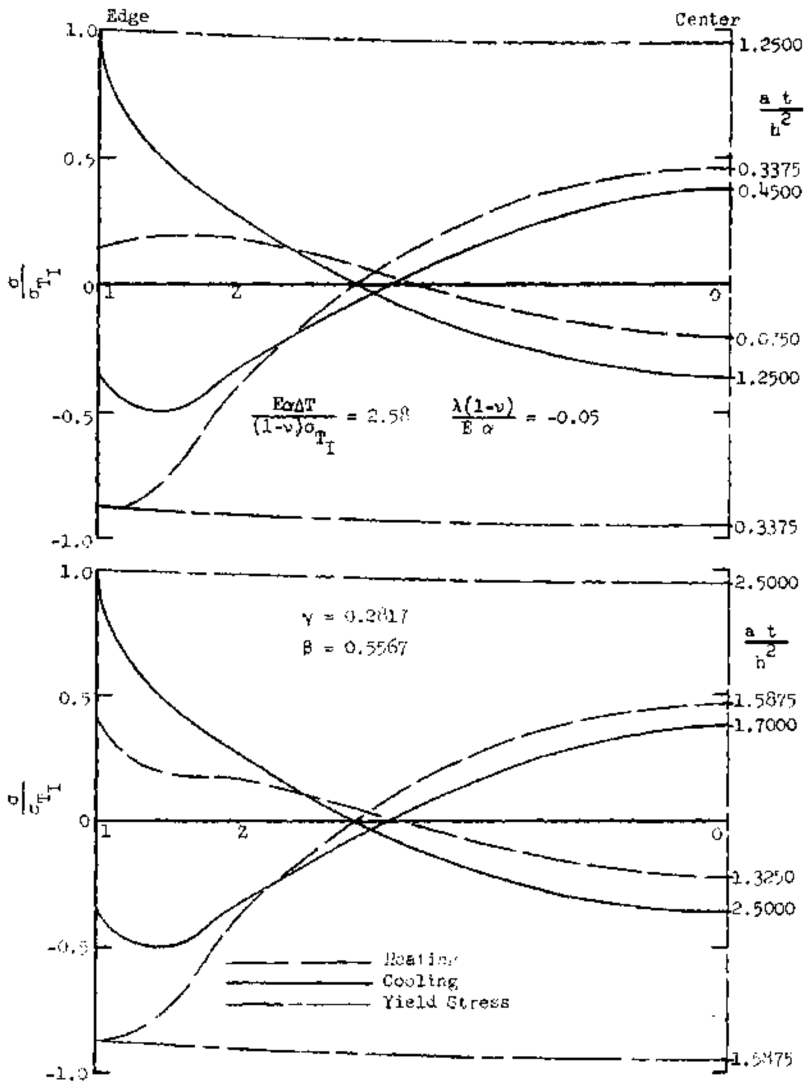


Figure 47. Stress Distributions for the First and Second Cycles Illustrating Shakedown Behavior of a Plate with the Yield Stress Linearly Decreasing with Temperature

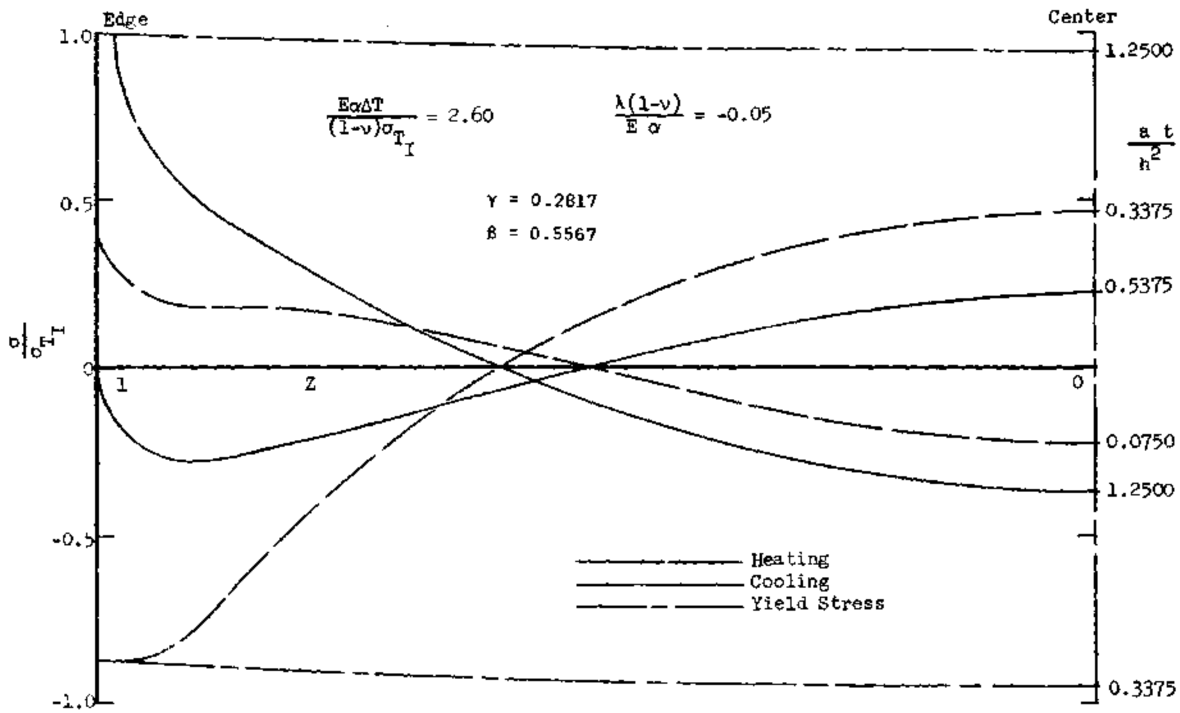


Figure 48. Stress Distributions for One Cycle Illustrating Alternate Plasticity Behavior of a Plate with the Yield Stress Linearly Decreasing with Temperature

APPENDIX C

COMPUTER PROGRAM

The computer program used to obtain the solution of the illustration problem in Appendix B is included in this appendix. The program is written in Algol-60 for the Univac 1108 computer at the Rich Electronic Computer Center at Georgia Tech. The symbols used in the program necessarily differed from those used elsewhere; therefore, the definitions of the program symbols are as follows:

I = Time increment.

J = Subinterval location.

TMIN = Minimum surface temperature.

TAVE = Average temperature of plate.

TDIF = (Surface temperature) - (Average temperature).

T(I,J) = $\frac{E\alpha(\text{Temperature})}{(1-\nu)\sigma_{T_m}}$ at the I time increment and the J subinterval location where σ_{T_m} is the yield stress at temperature TMIN.

S(J) = $\frac{\sigma}{\sigma_{T_m}}$ at the J subinterval location.

SYP(J) = $\frac{\sigma_{yp}}{\sigma_{T_m}}$ at the J subinterval location.

EEP(J) = $\frac{E(\text{Effective plastic strain})}{(1-\nu)\sigma_{T_m}}$ at the J subinterval location at time I.

EIP(J) = $\frac{E(\text{Effective plastic strain})}{(1-\nu)\sigma_{T_m}}$ at the J subinterval location at time I-1.

EP(J) $= \dfrac{E(\text{Plastic strain})}{(1-\nu)\alpha_{T_m}}$ at the J subinterval location at time I.

EIIP(J) $= \dfrac{E(\text{Plastic strain})}{(1-\nu)\alpha_{T_m}}$ at the J subinterval location at time I-1.

P $= \dfrac{2a\Delta t}{h^2 \Delta Z^2}$.

TEMP $= \dfrac{E\alpha(\Delta T)}{(1-\nu)\sigma_{T_m}}$.

N = Total number of subinterval points in the cross section.

M = Total number of time increments per temperature cycle.

M1 = Number of time increments in the heating portion of the temperature cycle.

D = Number of temperature cycles considered.

DELZ = Dimensionless subinterval distance (ΔZ).

ADELTH2 $= \dfrac{a\Delta t}{h^2}$

APERH2 $= \dfrac{a(\text{Period})}{h^2}$

SLOPE $= \dfrac{\lambda(1-\nu)}{E\alpha}$

AB = Either β or γ.

```
        BEGIN
COMMENT         THERMAL STRESSES IN FREE PLATE WITH PERIODICALLY VARYING
                TEMPERATURE ON BOTH SURFACES.
                YIELD STRESS MAY REMAIN CONSTANT OR DECREASE LINEARLY
                WITH TEMPERATURE.
                ELASTIC-PERFECTLY PLASTIC MATERIAL                       ;
REAL            B,F,P,TEMP,DELZ,ADELTH2,TAVE,TDIF,AB,APERH2,TIME1,SLOPE,
                TMIN                                                     ;
INTEGER         I,J,L,N,Z,H,M,R,M1,D                                     ;
REAL ARRAY      Y,X,EP,EEP,EIP,EIIP,S,SYP(1:11),         T(0:2000,1:12) ;
LIST            L1(I,TAVE,AB,TDIF,FOR J=1 STEP 1 UNTIL (N-((N-1)//2)) DO
                T(I,J),FOR J=1 STEP 1 UNTIL N-(N-1)//2 DO SYP(J),FOR J=1
                STEP 1 UNTIL N-(N-1)//2 DO S(J),FOR J=1 STEP 1 UNTIL
                N-(N-1)//2 DO EP(J))                                     ;
FORMAT          FMT1("I=",I4,X5, "TAVE=",D9.5, X6,"AB=",D9.5,X4,"TDIF=",
                D9.5, A1,"T(I,J)=",:N-(N-1)//2:(D9.5),A1,"SYP(J)=",
                :N-(N-1)//2:(D9.5),A1,"S(J)  =",:N-(N-1)//2:(D9.5),A1,
                "EP(J) =",:N-(N-1)//2:(D9.5),A1,1)                      ;
LIST            L2(TEMP,M,M1,P,APERH2,ADELTH2,SLOPE)                    ;
FORMAT          FMT2(E1,"TEMP=",D7.3,X2,"M=",I3,"STEPS/CYCLE",X2,"M1=",
                I3,"STEPS",X2,"P=",D5.3,"(LEQ1)",X2,"A*PERIOD/H**2=",D7.4
                ,X3,"A*DELTIME/H**2=",D10.8,"SLOPE(1-V)/(E.APH)=",D5.3,
                A3,1)                                                    ;
                N=21                                                     ;
                D=5                                                      ;
                DELZ= 2/(N-1)                                            ;
                APERH2 = 1.250                                           ;
                M=400                                                    ;
                ADELTH2 = APERH2/M                                       ;
                P= 2*  (ADELTH2/(DELZ * DELZ))                           ;
                FOR J = 1 STEP 1 UNTIL N-(N-3)//2 DO
                T(0,J)=2                                                 ;
                M1 = 108                                                 ;
                SLOPE = -.05                                             ;
COMMENT         PROCEDURE TO CALCULATE THE TEMPERATURE DISTRIBUTION
                THROUGH THE PLATE RESULTING FROM A PERIODICALLY VARYING
                TEMPERATURE ON BOTH SURFACES PRODUCED BY CHANGING THE
```

```
          SURFACE TEMPERATURE LINEARLY.                                        ;
          FOR TEMP = 10.100, 10.095, 10.090 DO
     BEGIN
          FOR I = 1 STEP 1 UNTIL M DO
     BEGIN
          IF I LEQ M1 THEN
          T(I,1) = TEMP * (I/M1) + T(0,1)                                      ;
          IF I GTR M1 THEN
          T(I,1) = TEMP * ((M-I)/(M-M1)) + T(0,1)                              ;
          FOR J = 2 STEP 1 UNTIL N-(N-1)//2 DO
          T(I,J)= P*((T(I-1,J-1)+T(I-1,J+1))/2) + (1-P)*T(I-1,J)               ;
          T(I,N-(N-3)//2) = T(I,N-(N+1)//2)                                    ;
     END                                                                       ;
          TMIN=T(M,1)                                                          ;
          FOR I = M+1 STEP 1 UNTIL (D   )*M DO
     BEGIN
          T(I,1)=T(I-M,1)                                                      ;
          FOR J = 2 STEP 1 UNTIL N-(N-1)//2 DO
          T(I,J)= P*((T(I-1,J-1)+T(I-1,J+1))/2) + (1-P)*T(I-1,J)               ;
          T(I,N-(N-3)//2) = T(I,N-(N+1)//2)                                    ;
     END                                                                       ;
          FOR I = (D-1)*M STEP -1 UNTIL 1    DO
          FOR J = 2 STEP 1 UNTIL N-(N-1)//2 DO
          T(I,J) = T(I+M,J)                                                    ;
          WRITE(FMT2,L2)                                                       ;
COMMENT   PROCEDURE TO NUMERICALLY CALCULATE THE THERMAL STRESSES
          AND PLASTIC STRAINS IN A FREE PLATE DUE TO THE ABOVE
          TEMPERATURE DISTRIBUTION.                                            ;
          FOR J = 2 STEP 1 UNTIL N-(N-1)//2 DO
     BEGIN
          EP(J)=0.0                                                            ;
          EEP(J)=0.0                                                           ;
          EIIP(J)=0.0                                                          ;
     END                                                                       ;
          FOR I=1 STEP 1 UNTIL (D*M) DO
     BEGIN
          B=0.0                                                                ;
```

```
          TAVE=0.0                                              ;
          FOR J = 1 STEP 1 UNTIL N-(N-1)//2 DO
     BEGIN
          IF J EQL 1 OR J EQL N-(N-1)//2 THEN
          B= T(I,J)/2 +B
          ELSE B= T(I,J)    +B                                  ;
          X(J)=0                                                ;
          Y(J)=0                                                ;
          SYP(J)=1+SLOPE*(T(I,J)-TMIN)                          ;
     END                                                        ;
LAB1:     Z=0                                                   ;
          F=0.0                                                 ;
          H=0                                                   ;
          FOR J = 1 STEP 1 UNTIL N-(N-1)//2 DO
          IF J EQL 1 OR J EQL N-(N-1)//2 THEN
          F= EP(J)/2 +F
          ELSE F= EP(J)+F                                       ;
          FOR J = 1 STEP 1 UNTIL N-(N-1)//2 DO
     BEGIN
          S(J)=2*(B/(N-1))- T(I,J)+2*(F/(N-1))- EP(J)           ;
          EIP(J)=EEP(J)                                         ;
          IF S(J) GTR SYP(J) THEN
     BEGIN
          EEP(J) = (S(J)-SYP(J)) + EEP(J)                       ;
          EP(J)=(EEP(J)/2)                                      ;
          X(J)=1                                                ;
     END                                                        ;
          IF S(J) LSS (-SYP(J)) THEN
     BEGIN
          EEP(J) = (S(J) + SYP(J)) + EEP(J)                     ;
          EP(J)=(EEP(J)/2)                                      ;
          Y(J)=1                                                ;
     END                                                        ;
          IF ABS(EEP(J)-EIP(J)) GTR 0.000001 THEN Z=Z+1         ;
     END                                                        ;
          IF Z GTR 0 THEN GO TO LAB1                            ;
          FOR J = 1 STEP 1 UNTIL N-(N-1)//2 DO
```

```
BEGIN
     IF (X(J) GTR 0 AND S(J) LSS SYP(J)) OR (Y(J) GTR 0 AND
     S(J) GTR (-SYP(J))) THEN
BEGIN
     EP(J)=EIIP(J)                                                 ;
     EEP(J)=2*EIIP(J)                                              ;
     X(J)=0                                                        ;
     Y(J)=0                                                        ;
     H=1                                                           ;
END  ELSE EIIP(J)=EP(J)                                            ;
END                                                                ;
     IF H GTR 0 THEN GO TO LAB1                                    ;
     IF I EQL 4*(I//4) THEN
BEGIN
     FOR J = 1 STEP 1 UNTIL N-(N-1)//2 DO
     IF J EQL 1 OR J EQL N-(N-1)//2 THEN
     TAVE = T(I,J)/(N-1) + TAVE
     ELSE TAVE = T(I,J)/(N-(N+1)//2) + TAVE                        ;
     AB=(TAVE - T(0,1))/TEMP                                       ;
     TDIF = T(I,1) - TAVE                                          ;
     WRITE(FMT1,L1)                                                ;
END                                                                ;
END                                                                ;
END                                                                ;
END PROGRAM                                                        ;
```

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VITA

Gerald Ware May was born in Warrenton, Georgia, on February 27, 1940, the son of Mr. and Mrs. Henry C. May. He attended elementary school in Warrenton and graduated from Richmond Academy, Augusta, Georgia, in 1958. Gerald attended Augusta Junior College from 1958 to 1960 where he was awarded a two-year pre-engineering degree. He attended Georgia Institute of Technology from 1960 to 1963 where he was awarded the degree Bachelor of Mechanical Engineering with honor in March, 1963. During the Fall of 1963 Gerald entered the Graduate Division of the Georgia Institute of Technology and began graduate work in the School of Engineering Mechanics. In June 1965 he received a M.S. in Engineering Mechanics. He continued his work at Georgia Tech in the School of Mechanical Engineering. In June, 1967, he received a M.S. in Mechanical Engineering and continued his work toward a Doctor of Philosophy in the School of Mechanical Engineering.

During the summer months of his undergraduate studies he worked as an engineering trainee for the E. I. du Pont de Nemours & Co. at the Savannah River Plant, Aiken, South Carolina, and upon graduating in 1963, he worked for the same company in Wilmington, Delaware, as a mechanical engineer in machine development in its Engineering Development Laboratory. During graduate studies he worked as a graduate teaching assistant, teaching undergraduate mechanics courses, and as a part-time instructor, teaching engineering graphics courses.

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